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O. Neugebauer **M. H. Stone** **O. Veblen**

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TABLE OF CONTENTS

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Mathematical Reviews

Vol. 8, No. 7 JULY-AUGUST, 1947 Pages 365-428

ALGEBRA

Kaplansky, Irving, and Riordan, John. The problème des ménages. Scripta Math. 12, 113–124 (1946).

A new proof is given of an explicit formula for the number of solutions of Lucas' problème des ménages [cf. Kaplansky, Bull. Amer. Math. Soc. 49, 784–785 (1943); these Rev. 5, 86] by means of a symbolic method [Kaplansky, ibid. 50, 906–914 (1944); these Rev. 6, 159]. The wider problem of the number of arrangements of n married couples at a round table, men and women alternating, such that r husbands are seated next to their own wives and $n-r$ are not, is also taken into account. Explicit formulas, recurrence relations and asymptotic expressions are obtained.

N. G. de Bruijn (Delft).

Fu, Chung-Sun. A problem on non-sensed circular permutations. Wu-Han Univ. J. Sci. 8, no. 1, 1.1–1.16 (1942).

The compact formula below is given for the number of non-sensed circular permutations of n objects of specification $(n_1n_2\cdots n_m)$, that is, n_1 of one kind, n_2 of a second, and so on:

$$(*)\qquad \tfrac{1}{2}\{h+n^{-1}\sum\varphi(d)[n_1d^{-1}, \cdots, n_md^{-1}]\},$$

where the summation is over all divisors, including 1 and g, of the greatest common divisor g of n_1 to n_m, $\varphi(d)$ is the number of numbers prime to and not greater than d, the bracketed m-variable function is the multinomial coefficient $(\sum n_id^{-1})![(n_1d^{-1})!\cdots(n_md^{-1})!]^{-1}$ and h is $[[n_1/2], \cdots, [n_m/2]]$ with interior brackets indicating integral parts, unless three or more of the n_i are odd, when it is zero. For n objects all unlike, specification (1^n), (*) reduces to the well-known result $(n-1)!/2$ [$g=1, h=0, n>2$].

J. Riordan (New York, N. Y.).

Mendelsohn, N. S. Symbolic solution of card matching problems. Bull. Amer. Math. Soc. 52, 918–924 (1946).

The following problem is considered. Let there be $a_1, \cdots, a_n$ cards, all considered different, of which a_r are marked r. It is required to find the number of arrangements of these cards in which none of the cards marked r appear in any of p_r specified places and also the number of arrangements in which these conditions are violated (1) exactly s times and (2) at most s times. Let p_{rs} be the number of places simultaneously forbidden to cards marked r or s; p_{rst} the number simultaneously forbidden to cards marked r, s or t, and so on. In the case in which all $p_{ij}=0$, the problem has been solved by Kaplansky [Bull. Amer. Math. Soc. 50, 906–914 (1944); these Rev. 6, 159]. A solution is obtained here for the general case in the form of a recurrence formula which gives the solution for n kinds of cards in terms of the solutions for less than n kinds. The procedure is also generalized to provide the solution of certain types of multiple matching problems.

T. N. E. Greville.

***Frazer, R. A., Duncan, W. J., and Collar, A. R. Elementary Matrices and Some Applications to Dynamics and Differential Equations.** Cambridge, at the University Press; New York, the Macmillan Company, 1946. xvi+416 pp. $4.00.

This is a reprint of the English edition of 1938. The first three chapters form a self-contained introduction to the algebra, calculus and classification, respectively, of matrices. The fourth chapter follows with a consideration of numerical methods for matrix calculations and the solution of algebraic equations of general degree. The next three chapters treat the solution of ordinary differential equations with constant coefficients and the numerical solution of those with variable coefficients.

After this preparatory purely mathematical half of the book, applications begin in chapter 8 with a general treatment of the dynamics of rigid and elastic bodies, including the motion of a rigid aeroplane. The next two chapters treat the small oscillations of linear dynamical conservative and dissipative systems, including iterative numerical methods of solution. The power of the latter methods is illustrated by a number of complicated systems, e.g., the torsional vibrations of a multicylinder engine. The last three chapters discuss problems in the field of special interest to the authors, namely aerodynamical systems with nonlinear frictional terms (solid friction). Here ordinary tests for stability fail and an instructive example is given of a system otherwise stable made unstable by the addition of solid friction. The theory of solid friction is then applied in detail to flutter problems, in chapter 12 to three aeroplanes of given characteristics, and in the final chapter to the pitching oscillations of a constrained airfoil. The latter is partly a discussion of experimental results not explained by the theory. Numerous examples completely worked out are given throughout the text. A bibliography of 51 items is included.

Although the authors are motivated by an interest in the behavior of the aeroplane, and consider explicitly only mechanical systems, the methods used apply equally well to other systems, for example, to electric circuits. The book is to be recommended to all engineers and applied mathematicians interested in matrix methods and/or dynamical systems, and of course particularly to those interested in aerodynamical computations. The treatment of matrix theory can also be recommended to the mathematical student not interested in applications. However, it is possible that those interested only in matrices as tools may find it hard to sustain their interest through the unbroken mathematical treatment with all applications deferred to the later chapters. The mathematical treatment is marred only by a failure to show that a complete solution to a system of linear differential equations with constant coefficients is afforded by the methods given. The treatment of canonical forms of matrices is also too brief. Otherwise the treatment is admirable, from the form of the proofs down to matters

of detail like, for example, in the discussion of finite rotations, the replacement of the more usual clumsy form of the Euler angles by those angles corresponding to roll, pitch and yaw, natural to describing an aeroplane, but just as convenient in discussing, say, the motion of a gyroscope.
L. C. Hutchinson (Brooklyn, N. Y.).

Price, G. B. Some identities in the theory of determinants. Amer. Math. Monthly **54**, 75–90 (1947).

In this expository paper the author considers the Binet-Cauchy multiplication theorem; Sylvester's theorem of 1839 and 1851; the Sylvester-Franke theorem; the Bazin-Reiss-Picquet theorem; the Cauchy, Jacobi, Franke and Reiss theorems; and Sylvester's theorem on super-determinants. Wherever possible the theorems are treated from the matrix standpoint and, thanks to a well chosen notation, the proofs are clear and easily read. The author concludes with a short historical account of the theorems and a detailed bibliography. *J. Williamson* (Flushing, N. Y.).

Chen, Teh-Chao. Note on the diagonalization of an impedance dyadic or matrix. Coll. Papers Sci. Engin. Nat. Univ. Amoy **1**, 43–54 (1943).

The diagonalization of an impedance matrix can be carried out in an infinite number of ways, some of the resulting forms being more practical than others. It would be desirable in certain cases for the result to yield the usual symmetrical components. The author (in dyadic terminology) applies a standard method of diagonalization based on the Cauchy-Hamilton identity to effect this end for 3-phase rotating machines and for symmetrical bilateral 3-phase networks, in contrast to previous diagonalizations not in this form [e.g., L. A. Pipes, Trans. Amer. Inst. Elec. Engrs. **60**, 351–356 (1941); these Rev. **3**, 256].
L. C. Hutchinson (Brooklyn, N. Y.).

Cherubino, Salvatore. Sulle condizioni di esistenza di una matrice di Riemann e sui moduli delle curve algebriche. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **3**, 98–105 (1942).

A matrix w with p rows and $2p$ columns of complex numbers is a Riemann matrix if there exists a rational skew matrix C such that $wCw'=0$, $iwC\bar{w}'$ is positive definite. Conditions on a prescribed w that C shall exist were given in terms of minors by G. Scorza in 1913 and are obtained here in a slightly more symmetrical form. *A. A. Albert.*

Šestakov, V. I. Algebra of two-pole circuits, constructed exclusively from two-pole switches (algebra of A-circuits). Avtomatika i Telemehanika **1941**, no. 2, 15–24 (1941). (Russian)

Cf. Uchenye Zapiski Moskov. Gos. Univ. Matematika **73**, 45–48 (1944); these Rev. **7**, 359.

Abstract Algebra

Lalan, Victor. L'anneau de Boole à seize éléments et le calcul des propositions. C. R. Acad. Sci. Paris **224**, 432–434 (1947).

The free Boolean algebra with two generators is discussed. The 4 primes and 6 elementary logical functions are enumerated and interpreted logistically. *G. Birkhoff.*

Schützenberger, Maurice Paul. Remarques sur la notion de clivage dans les structures algébriques et son application aux treillis. C. R. Acad. Sci. Paris **224**, 512–514 (1947).

The author introduces various concepts into universal algebra, relevant to the lattice $U(S)$ of all families of identities satisfied in an algebraic system [cf. G. Birkhoff, Proc. Cambridge Philos. Soc. **31**, 433–454 (1935)]. Especially, he exhibits "cleavages" [in the sense of P. Whitman, Amer. J. Math. **65**, 179–196 (1943); these Rev. **4**, 129] in $U(S)$, with specific examples. *G. Birkhoff* (Cambridge, Mass.).

Rocco Boselli, Anna. Equazioni algebriche nelle algebre del 4° ordine dotate di modulo. Rend. Accad. Sci. Fis. Mat. Napoli (4) **12**, 363–385 (1942).

The paper is concerned with the number of roots of algebraic equations in certain algebras of order four with a unity quantity and over the complex field. The results are applied to count the number of nilpotent quantities. None of the known structure theory of algebras is used and the results are obsolete. *A. A. Albert* (Rio de Janeiro).

Krasner, Marc. Théorie non abélienne des corps de classes pour les extensions finies et séparables des corps valués complets: approximation des corps valués complets par les suites de corps valués complets. C. R. Acad. Sci. Paris **224**, 173–175 (1947).

Krasner, Marc. Théorie non abélienne des corps de classes pour les extensions finies et séparables des corps valués complets: approximation des corps de caractéristique $p\neq 0$ par ceux de caractéristique 0; modifications de la théorie. C. R. Acad. Sci. Paris **224**, 434–436 (1947).

If k is a field with a valuation, an equivalence relation Π in k is called a multiplicative divisor if $\alpha\equiv\beta$, Π, and $|\alpha'-\beta'|/\max(|\alpha'|, |\beta'|)\leqq|\alpha-\beta|/\max(|\alpha|, |\beta|)$ imply $\alpha'\equiv\beta'$, Π. The norm $|\Pi|$ of Π is the upper bound on the semi-real axis [Krasner, same C. R. **219**, 433–435 (1944); these Rev. **7**, 364] of $|\alpha-\beta|/\max(|\alpha|, |\beta|)$, where $\alpha\equiv\beta$, Π. Sums and products in the quotient set k/Π are defined in the usual way. If Π, Π' are multiplicative divisors in fields k, k' and $|\Pi|=|\Pi'|$, an isomorphism ζ of k/Π on k'/Π' is called a residual isomorphism of k on k' of norm $|\zeta|=|\Pi|$. Now if k is a complete field, $\{k_m\}$ a sequence of such fields and $\zeta=\{\zeta_m\}$ a sequence of residual isomorphisms of k on k' such that $|\zeta_m|\to 0^+$ as $m\to\infty^-$, the sequence $\{k_m\}$ is called an approximating sequence to k. If K is a finite separable extension of k the residual automorphisms and approximating sequence can be extended to give an approximating sequence $\{K_m\}$ to K. The integral domain, ideals, discriminant, etc., of K/k are limits of those of K_m/k_m.

Every complete field of characteristic $p\neq 0$ can be approximated in the above sense by a sequence of complete fields of characteristic zero. This result can be used to prove the main theorems of the class field theory for separable Abelian extensions of locally compact fields of characteristic $p\neq 0$, starting from the corresponding theorems in the ordinary local class field theory. Some modifications of the author's previously developed theory [same C. R. **220**, 28–30, 761–763 (1945); these Rev. **7**, 364] are discussed.
D. C. Murdoch (Vancouver, B. C.).

THEORY OF GROUPS

Artin, E. Theory of braids. Ann. of Math. (2) **48**, 101–126 (1947).

In his original paper on braids [Abh. Math. Sem. Hamburgischen Univ. **4**, 47–72 (1926)] the author established the fundamental facts of the theory by a mixture of algebraic and geometric arguments with frequent appeal to intuition. The braids were studied by means of their projections. In the present paper the method of projections is replaced by the method of braid coordinates. An n-braid B is defined as a set of n disjoint curves (called the strings of the braid) in three-space such that each plane parallel to the (x, y)-plane intersects each curve in exactly one point. Moreover it is assumed that the strings become parallel to the z-axis for $|z|$ sufficiently large. A coordinate system $(\bar{x}, \bar{y}, \bar{z})$ in three-space is called a system of braid coordinates for B if (1) $\bar{z}=z$ is the Cartesian z-coordinate, (2) $\bar{x}$ and $\bar{y}$ are constant along each braid string, (3) $\bar{x}=x$, $\bar{y}=y$ for large values of $x^2+y^2+z^2$. The existence of braid coordinates for each braid is established and is a fundamental result. Several concepts of isotopy of braids are introduced and their equivalence is shown using braid coordinates. The isotopy classes of n-braids form a groupoid; by suitably fixing the initial and terminal values for each braid, groups are obtained.

Given a braid B, select numbers z^+ and z^- such that the braid strings are constant for $z \geqq z^+$ and $z \leqq z^-$. The planes $z=z^+$ and $z=z^-$ are then punctured by B in n points and the fundamental groups F^+ and F^- of the punctured planes are free groups with n generators. A definite procedure for describing these generators is introduced. Following the braid coordinates induces a homeomorphism of the punctured plane $z=z^-$ onto the punctured plane $z=z^+$ and thus yields an isomorphism $\phi_B: F^- \approx F^+$. The second fundamental result is that ϕ_B determines the isotopy class of B. Each braid group is thus represented faithfully as a group of automorphisms of a free group. The remaining results are largely algebraic. Among others an algebraic characterization of the isomorphisms ϕ_B is given. *S. Eilenberg.*

Bohnenblust, F. The algebraical braid group. Ann. of Math. (2) **48**, 127–136 (1947).

In the paper reviewed above, Artin has established generators $\sigma_1, \cdots, \sigma_{n-1}$ for the group of isotopy classes of n-braids with fixed initial and terminal values. He also proved the relations

$$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad i=1, \cdots, n-2,$$
$$\sigma_i\sigma_j=\sigma_j\sigma_i, \qquad i<j-1,$$

without proving that these are the defining relations. This is done in this paper in a purely algebraic way, by considering the group given by $\sigma_1, \cdots, \sigma_{n-1}$ and the above relations and representing it faithfully as a group of suitable automorphisms of a free group with n generators. Except for details of exposition the results seem to be contained in those of A. Markoff [Foundations of the algebraic theory of tresses, Trav. Inst. Math. Stekloff **16** (1945); these Rev. **8**, 131]. *S. Eilenberg* (New York, N. Y.).

Kaloujnine, Léo. Sur les p-groupes de Sylow du groupe symétrique de degré p^m. (Sous-groupes caractéristiques, sous-groupes parallélotopiques). C. R. Acad. Sci. Paris **224**, 253–255 (1947).

In previous reports [same C. R. **221**, 222–224 (1945); **222**, 1424–1425 (1946); **223**, 703–705 (1946); these Rev. **7**, 239; **8**, 13, 251] the author has studied the Sylow subgroups P_m of the symmetric group of degree p^m. In this note the characteristic subgroups of P_m are discussed. Let T be the automorphism of P_m which sends the element $[a(x_1, \cdots, x_{s-1})]$ into the element $[t_s a(x_1, \cdots, x_{s-1})]$, where $t_s \neq 0$ is an element of $GF(p)$, $s=1, \cdots, m$. [Cf. the previous reviews for this notation.] The automorphisms of this type form an Abelian subgroup T_m of the full automorphism group of P_m. It is shown that a given subgroup of P_m is characteristic if and only if it admits T_m. It follows readily that every characteristic subgroup belongs to a wider class of subgroups ("sous-groupes parallélotopiques") whose structure may be described in terms of the degrees of the polynomials appearing in the representation of its elements. It is also shown that with every element of P_m there is associated a vector which remains invariant under all automorphisms. *D. G. Duncan* (Vancouver, B. C.).

***Zappa, Guido. Sulla costruzione dei gruppi prodotto di due dati sottogruppi permutabili tra loro.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 119–125. Edizioni Cremonense, Rome, 1942.

Suppose that the group G contains subgroups A, B with the following property: (1) every element in G may be represented in one and only one way in the form ab for a in A and b in B. Then every element in G may also be represented in one and only one way in the form $b'a'$ for a' in A and b' in B. Thus there exist, to given elements a and b in A and B, respectively, uniquely determined elements a^b in A and ${}^ab$ in B such that (2) $ab={}^ab a^b$. If the group H contains A and B and satisfies condition (1) also, and if H leads to the same functions ${}^ab$ and a^b as does G, then there exists an isomorphism of G upon H which leaves invariant every element in A and every element in B. If groups A and B and mappings ${}^ab$ and a^b are given, then the following conditions are necessary and sufficient for the existence of a group G satisfying (1) and (2): ${}^ab$ is, for every given a in A, a permutation of the elements in B; a^b is, for every given b in B, a permutation of the elements in A; $(a^b)^{b'}=a^{bb'}$; ${}^{a'}({}^ab)={}^{a'a}b$; $(aa')^b=a^{b^*}a'^b$ with $b^*={}^{a'}b$; ${}^a(bb')={}^ab\,{}^{a^*}b'$ with $a^*=a^b$. *R. Baer* (Urbana, Ill.).

Golberg, P. The Silov Π-groups of locally normal groups. Rec. Math. [Mat. Sbornik] N.S. **19**(61), 451–460 (1946). (Russian. English summary)

The author defines locally normal groups (every finite subset generates a finite invariant subgroup), locally soluble groups (every finite subset generates a finite soluble subgroup) and local inner automorphisms (operating innerly on every finite subset), in order to generalize the theorems of P. Hall relating to characteristic properties of finite soluble groups and Sylow systems [J. London Math. Soc. **3**, 98–105 (1928); **12**, 198–200 (1937); Proc. London Math. Soc. (2) **43**, 316–323 (1937)]. *H. Freudenthal.*

Eilenberg, Samuel, and MacLane, Saunders. Cohomology theory in abstract groups. I. Ann. of Math. (2) **48**, 51–78 (1947).

Let Π and G be groups, G being Abelian and additively written. Assume that Π acts as a group of left operators on G. A k-dimensional cochain, coefficients G, in Π is any function $f(x_0, \cdots, x_k)$ with values in G and satisfying the homogeneity condition $f(xx_0, \cdots, xx_k)=xf(x_0, \cdots, x_k)$. The usual definition of coboundary gives a homogeneous $(k+1)$-

cochain δf and leads to the definition of homology groups $H^n(\Pi, G)$. (For the proofs of theorems, the authors prefer an equivalent nonhomogeneous approach.) Main theorems:

$$H^n(\Pi, C(\Pi, G)) \cong H^{n+1}(\Pi, G),$$
$$H^n(\Pi, \mathrm{Hom}\,(R, G)) \cong H^{n+2}(\Pi, G).$$

Here C is the additive group of functions $h(x)$ on Π to G with $h(1)=0$; R is the group of relations in a representation $\Pi = F/R$ (F free); Hom (R, G) is the additive group of homomorphisms $R \to G$; Π acts, of course, as a group of operators on C and Hom (R, G) in a specified manner.

P. A. Smith (New York, N. Y.).

Koszul, Jean-Louis. Sur le troisième nombre de Betti des espaces de groupes de Lie compacts. C. R. Acad. Sci. Paris **224**, 251–253 (1947).

The theorem is that the third Betti number of a simple compact Lie group is one. (This had previously remained unproved for certain of the exceptional groups.) The proof is sketched. It is based on a consideration of the Grassmann algebra L of left-invariant differential forms; a scalar product is defined in L by means of an invariant Riemannian metric. *P. A. Smith* (New York, N. Y.).

***Følner, Erling. Almost periodic functions on Abelian groups.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 356–362. Jul. Gjellerups Forlag, Copenhagen, 1947.

No proofs are indicated. A set E on an Abelian group G shall be termed relatively dense if there exists a finite number of elements $a_1, \cdots, a_k$ in G, such that, for any element t in G, one of the differences $t-a_1, t-a_2, \cdots, t-a_k$ lies in E. If also the group is countable and has discrete topology, then the author asserts that there exist characters $\chi_1(x), \cdots, \chi_q(x)$, with $q=k^2$ (thus q depending only on k), such that every element x of G which satisfies simultaneously the q inequalities $\Re\{\chi_p(x)\}>0$, $p=1, \cdots, q$, can be written in the form $x=\tau_1+\tau_2+\tau_3+\tau_4-\tau_5-\tau_6-\tau_7-\tau_8$, with the τ's from E. This theorem is patterned closely on a theorem of Bogolioùboff [Ann. Chaire Phys. Math. Kiev 4, 185–205 (1939)] for the straight line $-\infty<x<\infty$. As in Bogolioùboff's case it can be used to prove the completeness of characters on any Abelian group. *S. Bochner*.

Levi, F. W. On semigroups. II. Bull. Calcutta Math. Soc. **38**, 123–124 (1946).

Corresponding to any nonassociative multiplicative system W there exists a semigroup $S(W)$ uniquely determined up to isomorphism, such that every semigroup which is homomorphic to W is homomorphic to $S(W)$. Considering next homomorphic mappings of a semigroup S on a group G the author shows that every such homomorphism is uniquely determined by a complete normal subsemigroup N of S which is the kernel of the homomorphism. Conversely every such subsemigroup determines a unique homomorphism. A subsemigroup N of S is complete if every element of S is a left divisor of some element of N. The definition of normality is that of the author's previous paper [Bull. Calcutta Math. Soc. 36, 141–146 (1944); these Rev. 6, 202]. *D. C. Murdoch* (Vancouver, B. C.).

Krasner, Marc, et Kuntzmann, Jean. Remarques sur les hypergroupes. C. R. Acad. Sci. Paris **224**, 525–527 (1947).

Attention is drawn to two minor errors in a previous paper [Krasner, Duke Math. J. 6, 120–140 (1940); these Rev. 1, 260]. It is shown that neither of these errors invalidates the main results of that paper. *D. C. Murdoch*.

NUMBER THEORY

Uhler, H. S. On Mersenne's number M_{199} and Lucas's sequences. Bull. Amer. Math. Soc. **53**, 163–164 (1947).

The author announces that $2^{199}-1$ is composite. This is the result of applying Lucas's test sequence 3, 7, 47, 2207, $\cdots$. The 198th term of this sequence was found to be congruent to 8387 51186 96313 46717 54322 73509 44243 96183 21834 95333 72125 49353 modulo $2^{199}-1$. However, no factor of $2^{199}-1$ is revealed. There remain only $2^{193}-1$ and $2^{227}-1$ to test in order to complete the investigation of the truth of Mersenne's statement of 1644. *D. H. Lehmer*.

Ljunggren, Wilhelm. On a theorem of R. Tambs Lyche. Norske Vid. Selsk. Forh., Trondhjem **17**, no. 28, 110–113 (1944).

A short proof is given of the following theorem of Tambs Lyche [Avh. Norske Vid. Akad. Oslo I **1944**, no. 9 (1945); these Rev. 7, 505]. Let ${}_nS_k$ be the elementary symmetric function of degree k on the n integers $1, 3, \cdots, 2n-1$ and let ${}_nC_k$ be the binomial coefficient. Then ${}_nS_k$ is exactly divisible by the same power of 2 as ${}_nC_k$ if k is even, and as $n({}_nC_k)$ if k is odd. *I. Niven* (West Lafayette, Ind.).

Ljunggren, Wilhelm. A theorem on the elementary symmetric functions of the n first odd numbers. Norske Vid. Selsk. Forh., Trondhjem **19**, no. 5, 14–17 (1946).

Define $\beta_0=1$, $\beta_1=-\frac{1}{2}$, $\beta_{2t}=(-1)^{t-1}B_t$, $\beta_{2t+1}=0$ $(t\geqq 1)$, where the B_t are Bernoulli's numbers. The following extension of Tambs Lyche's theorem [cf. the preceding review] is proved:

$$({}_nS_{2q})/({}_nC_{2q}) \equiv ({}_nS_{2q+1})/\{n^2({}_{n-1}C_{2q})\} \equiv (1+4\beta)^{2q} \pmod{2^r},$$

where β^s is to be replaced by β_s and 2^r is the highest power of 2 dividing n. The proof makes use of a result of Frobenius on Bernoulli's numbers. A similar extension is also made of a result of E. Jacobsthal. *I. Niven*.

Ljunggren, Wilhelm. Sur la solution de quelques équations diophantiennes biquadratiques à deux inconnues. Norske Vid. Selsk. Forh., Trondhjem **16**, no. 28, 103–105 (1944).

The equations $(ax^2+bx+c)^2-Dy^2=-N$ and

$$x^2-D(ay^2+by+c)^2=N$$

with integral coefficients, $a>0$, $N>0$ and $D>1$, are considered. By a p-adic method of T. Skolem [C. R. Huitième Congrès Math. Scandinaves (Stockholm, 1934), Lund, 1935, pp. 163–188] it is proved that there is an upper bound, which can be determined, to the number of integral solutions x and y. *I. Niven* (West Lafayette, Ind.).

***Ljunggren, Wilhelm. A Diophantine equation with two unknowns.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 265–270. Jul. Gjellerups Forlag, Copenhagen, 1947.

The Diophantine equation $x^2-Dy^{2n}=1$ with $n>3$ and $D+1$ not a square has at most 2 solutions in positive integers x and y. These solutions can be found readily if the

fundamental unit of $R(\sqrt{D})$ is known. If $n=3$, the result holds for all D. The result also holds when $D+1$ is a square if D exceeds a certain limit depending only on n. The proof employs work of Siegel [Math. Ann. **114**, 57–68 (1937)] on the equation $ax^n-by^n=c$ and Tartakowsky [Bull. Acad. Sci. URSS [Izvestia Akad. Nauk SSSR] (6) **20**, 301–324 (1926)] on the equation $x^{2n}-Dy^{2n}=1$. *I. Niven.*

Bell, E. T. Mahavira's Diophantine system. Bull. Calcutta Math. Soc. **38**, 121–122 (1946).

The complete integer solution is given for a system noted by B. Datta in his account of Mahavira's work in Diophantine analysis [Bull. Calcutta Math. Soc. **20**, 267–294 (1930), in particular, pp. 283–288]. This system, consisting of the equations $m(x+y)=n(u+v)$, $pxy=quv$, where m, n, p, q are any constant integers and x, y, u, v are the unknowns, is separable [E. T. Bell, Trans. Amer. Math. Soc. **57**, 86–101 (1945); these Rev. **6**, 256]. The regular and singular solutions which make up the complete integer solution fall into sets according to the divisors of m, n, p, q.

W. H. Gage (Vancouver, B. C.).

***Cornacchia, Giuseppe. Sulle legge di formazione e sulle proprietà dei quozienti incompleti dello sviluppo di $\sqrt{q}$ in frazione continua in dipendenza dei valori h assunti dalla forma x^2-qy^2 per i termini delle ridotte—Applicaz. alla risoluz. dell'equaz. $x^2\pm hy^2=q$.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 999–1004. Edizioni Cremonense, Rome, 1942.

Bambah, R. P. On complete primitive residue sets. Bull. Calcutta Math. Soc. **38**, 113–116 (1946).

The author constructs two complete sets of odd residues $r_1, \cdots, r_{2^{m-1}}$; $s_1, \cdots, s_{2^{m-1}}$, mod 2^m, $m\geqq 3$, such that $r_1s_1, \cdots, r_{2^{m-1}}s_{2^{m-1}}$ is again a complete set of odd residues mod 2^m. There are exactly 2^{m-3} such sets of r's and s's obtainable by his method. *H. B. Mann.*

Gillis, J. Sequences of positive integers. J. London Math. Soc. **21**, 93–98 (1946).

Let M be a sequence of positive integers, $\delta^*(M)$, $\delta_*(M)$ the upper and lower asymptotic densities of M. Let $\mathfrak{M}=\{M_n\}$ be an infinite class of sets M_n of positive integers, $D_{n_1,\cdots,n_\lambda}$ the intersection of $M_{n_1}, \cdots, M_{n_\lambda}$ and $\delta_*(M_n)=\mu_n$. The following theorems are proved. (1) If, for some positive λ, $\liminf_{k\to\infty} k^{1-1/\lambda}/\sum_{n=1}^k\mu_n=0$, then if $\epsilon>0$ there exist numbers $n_1, \cdots, n_\lambda$ such that $\delta^*(D_{n_1,\cdots,n_\lambda})>\mu_{n_1}\cdots\mu_{n_\lambda}(1-\epsilon)$. (2) If $\mu_n>\mu>0$ for all n and if $\epsilon>0$ and λ is any integer we can find an infinite subclass Δ of $\mathfrak{M}$ such that, if $M_{n_1}, \cdots, M_{n_\lambda}$ are in Δ, then $\delta^*(D_{n_1,\cdots,n_\lambda})>\mu^\lambda-\epsilon$. (3) If $\{\epsilon_n\}$ is any sequence of positive numbers and $\mu_n>\mu>0$ for all n then we can find an infinite subclass $\{M_{n_k}\}$ of $\mathfrak{M}$ such that, for any integer λ, if $p_1, \cdots, p_\lambda$ are any values of n_k with $k>\lambda$ then $\delta^*(D_{p_1,\cdots,p_\lambda})>\mu^\lambda-\epsilon_\lambda$. The last two statements are proved assuming the axiom of choice. *H. B. Mann.*

Roussel, André. Sur le rattachement de certaines questions d'arithmétique à un principe d'extremum. Ann. Sci. École Norm. Sup. (3) **63**, 45–79 (1946).

Full details are given of the following two theorems which, with outlines of the proofs, were announced earlier [C. R. Acad. Sci. Paris **217**, 496–497 (1943); these Rev. **6**, 118]: p is a prime if and only if

$$\int^{p-1}(\cos \pi x \cos (\pi p/x))^{2m}dx<(1-1/(2p^6))^m,$$

where m has any value not less than $4p^6 \log (8p^7)$; r and s are relatively prime, with $s>r$, say, if and only if

$$\int_2^{s^s}(\cos \pi x \cos (\pi r/x) \cos (\pi s/x))^{2m}dx<(1-1/(2s^6))^m,$$

where $m\geqq 30s^6 \log (70s^7)$; this is generalized to give a necessary and sufficient condition that n integers are relatively prime. The proofs make use of the theorem of Riesz (a proof of which is given) that the maximum of a continuous function $f(x)$ in a closed finite interval (a, b) is given by $\lim_{m\to\infty} (\int_a^b(f(x))^m dx)^{1/m}$. Furthermore, defining

$$S_m=\sum_{k=2}^{p-1}(\cos (\pi p/k))^{2m},$$

the author proves that p is a prime if and only if $S_m<(1-(2/p^2))^m$, with $m>\frac{1}{2}(p^2-4) \log (p-2)$. Again, p is a prime if and only if $T_m<4^mp^{2m}$, with $m>12p^4 \log (p-2)$, T_m being $\sum_{j,k=2}^{p-1}(4p^2-(p/j-k)^2)^m$. This latter condition is converted to one involving a (double) contour integral, as is also the first result mentioned in this review.

I. Niven (West Lafayette, Ind.).

Kesava Menon, P. Some congruence properties of the φ-function. Proc. Indian Acad. Sci., Sect. A. **24**, 443–447 (1946).

This paper gives the proofs of four theorems on divisibility properties of $\phi(a^n-b^n)$, where $1<b<a$ are coprime integers and ϕ is Euler's totient function. Theorem 1 asserts that $\phi(a^n-b^n)$ is divisible by n. Theorem 4 extends this as follows. Let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ be the canonical factorization of n in which the distinct primes p_i are ordered so that $p_{i-1}>p_i$. Let $s_i=\alpha_1+\alpha_2+\cdots+\alpha_{i-1}$, $2t_i=\alpha_i+\alpha_i^2$. Then $\phi(a^n-b^n)$ is divisible by $\prod_{i=1}^r p_i^{\alpha_i s_i+t_i}$. *D. H. Lehmer.*

Mahler, Kurt. On reduced positive definite quaternary quadratic forms. Nieuw Arch. Wiskunde (2) **22**, 207–212 (1946).

According to Minkowski's definition, a positive definite n-ary form $f(x)=\sum\sum a_{hk}x_hx_k$ is said to be reduced if $f(x)\geqq a_{hh}$ for all integers $x_1, \cdots, x_n$ such that $(x_h, x_{h+1}, \cdots, x_n)=1$ (this holding for each value of h) and also $a_{12}\geqq 0$, $a_{23}\geqq 0$, $\cdots$, $a_{n-1,n}\geqq 0$. Using a classical theorem of Korkine and Zolotareff [Math. Ann. **5**, 581–583 (1872)] on the extreme form $x_1^2+x_2^2+x_3^2+x_4^2+(x_1+x_2+x_3)x_4$, the author proves that every reduced quaternary form of determinant D satisfies $a_{11}a_{22}a_{33}a_{44}\leqq 4D$. *H. S. M. Coxeter* (Toronto, Ont.).

Mordell, L. J. Lattice points in some n-dimensional non-convex regions. I, II. Nederl. Akad. Wetensch., Proc. **49**, 773–781, 782–792 = Indagationes Math. **8**, 476–484, 485–495 (1946).

(A) Let R be the n-dimensional region defined by $|x_n|\leqq f(|x_1|, \cdots, |x_{n-1}|)$, where (for $x_1>0, \cdots, x_{n-1}>0$) $f\geqq 0$ is steadily decreasing and differentiable and $\partial f/\partial x_i$ are steadily increasing functions of each variable separately. Let $x_1/p_1+x_2/p_2+\cdots+x_n/p_n=1$ be the tangent plane to $x_n=f(x_1, \cdots, x_{n-1})$ at a point $(\xi_1, \cdots, \xi_n)$, $\xi_i>0$; we suppose that $p_n\leqq 2\xi_n$ and that $f(x_1, \cdots, x_{n-1})/p_n\geqq 1-x_1/p_1-\cdots-x_{n-1}/p_{n-1}$ for all $x_i>0$ [this condition is not stated in the paper, but it is necessary in order to ensure the correctness of the conclusion on p. 486 = 783, line 3 from above; in the case $n=2$, it is a consequence of the other conditions]. Denote by V_n the volume of the region $x_1/p_1+\cdots+x_n/p_n\geqq\frac{1}{2}$,

$x_n+\frac{1}{2}p_n\leqq f(x_1, \cdots, x_{n-1})$, $x_1\geqq\xi_1, \cdots, x_{n-1}\geqq\xi_{n-1}$, $x_n\geqq 0$. Let Λ be an n-dimensional lattice of determinant Δ, where $0<\Delta\leqq(p_1 \cdots p_n)/n!+2^nV_n$. Then R contains a lattice point other than the origin. (B) Results analogous to (A); however, the x_i's are no longer coordinates, but sums of coordinates $l_1, l_2, \cdots$: $x_1=l_1+\cdots+l_\alpha$, $x_2=l_{\alpha+1}+\cdots+l_{\alpha+\beta}, \cdots$. [The denominator on p. 488=785, line 9 from below, and p. 489=786, line 6 from above, should be $n!$ instead of $\alpha!\beta!\cdots$.]

Applications. (C) Simultaneous approximation. Proof of a result of Blichfeldt [Trans. Amer. Math. Soc. 15, 227–235 (1914)] in the form of an integral and a better evaluation, due to Davenport, of this integral. (D) Proof of a theorem of Koksma and Meulenbeld [Nederl. Akad. Wetensch., Proc. 45, 256–262, 354–359, 471–478, 578–584 (1942); these Rev. 5, 256]. (E) Evaluation of $|x_1|^\lambda+\cdots+|x_n|^\lambda$ $(0<\lambda<1)$ and of the symmetrical function $\sum|x_1x_2\cdots x_r|$. [Concerning pp. 491–492=788–789: in order to ensure the correctness of these considerations, we must suppose that the least value of $f(x_1, \cdots, x_{n-1})$, the value of the sum $x_1+\cdots+x_{n-1}$ being given, is taken at the point $x_1=x_2=\cdots=x_{n-1}$. This condition is satisfied in the applications given in the paper. In the formula for V_2 on p. 495=792 the factor $(n-1)^{r-1}$ is omitted in the first term of the curly bracket.]

V. Jarník (Prague).

Cohn, Harvey. Note on almost-algebraic numbers. Bull. Amer. Math. Soc. 52, 1042–1045 (1946).

Consider a class of power series $\sigma(x)=x^{e_0}+a_1x^{e_1}+a_2x^{e_2}+\cdots$, where $a_1=r_1/S_1$, $a_2=r_2/S_2, \cdots$ are rational coefficients different from zero and $e_0, e_1, e_2, \cdots$ denote rapidly increasing integers. For any positive integer h let g_h be the maximum of the numbers $1, |a_1|, \cdots, |a_h|$ and let d_h be the least common multiple of the denominators $S_1, \cdots, S_h$. Then it is shown that $\sigma(x)$ has a transcendental value for every algebraic number $x\neq 0$ within its circle of convergence if the following three conditions are fulfilled: $\lim_{h\to\infty} e_{h+1}/e_h=\infty$, $\lim_{h\to\infty} e_{h+1}/\log g_h=\infty$, $\lim_{h\to\infty} e_{h+1}/\log d_h=\infty$.

J. Popken (Utrecht).

Coxeter, H. S. M. Integral Cayley numbers. Duke Math. J. 13, 561–578 (1946).

The multiplication table of the Cayley algebra, with basal units $e_0=1, e_1, \cdots, e_7$, may be written concisely as $e_r^2=e_re_{r+1}e_{r+3}=-1$, $r=1, \cdots, 7$, with $e_{r+7}=e_r$ and with "$e_1e_2e_4=-1$," etc., interpreted to mean $e_2e_4=-e_4e_2=e_1$, $e_4e_1=-e_1e_4=e_2$, $e_1e_2=-e_2e_1=e_4$, etc., like the quaternion relations $i^2=j^2=k^2=ijk=-1$. This is one of 480 possible ways of numbering $e_1, \cdots, e_7$ and choosing signs, which was also used, in studies of elliptic 7-space, by Cartan and Schouten [Nederl. Akad. Wetensch., Proc. 29, 933–946 (1926)]. The 7 triads above, and also those with a repeated element, are associative, while the remaining 28 triads are anti-associative, for example, $e_1e_2\cdot e_3=-e_6=-e_1\cdot e_2e_3$. The conjugate of $a=a_0+\sum a_re_r$ is $\bar{a}=a_0-\sum a_re_r$, a_0 and a_r real; the norm $N(a)=a\bar{a}=\bar{a}a=a_0^2+\sum a_r^2$. The author proves that $N(a)N(b)=N(ab)$, which is the 8-square theorem. He employs Dickson's definition of an integral set (maximal order) J, based on the rank equation $x^2-2a_0x+N(a)=0$ of a, with a_0 and a_r rational. He shows that there are exactly 7 distinct J's which contain $e_1, \cdots, e_7$. Kirmse [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math. Phys. Kl. 76, 63–82 (1925)] had erroneously concluded that there were 8, while Dickson [J. Math. Pures Appl. (9) 2, 281–326 (1923)] had correctly concluded that there are 3 which contain certain quaternion units besides $e_1, \cdots, e_7$. The author exhibits a basis, for a J, which has special metric properties expressible in terms of the distances between the nodes of a certain tree in Euclidean 8-space E_8. In E_8 the 240 units ($a\epsilon J$, $N(a)=1$) of J form the vertices of a polytope 4_{21} [see the author's forthcoming book, "Regular Polytopes"], while J is the honeycomb 5_{21}. The maximality of J follows from considerations involving the closest packing of spheres. The author determines, in compact form, the 2160, 6720 and 17280 elements of J of norm 2, 3 and 4, respectively, and shows their geometrical significance. Some of those of norm 2 and 4, along with the units, play a special rôle in generating the ideals of the Cayley-Dickson algebra [Mahler, Proc. Roy. Irish Acad. Sect. A. 48, 123–133 (1942); these Rev. 4, 185], but it is not known which ones, or what their geometrical significance may be.

R. Hull.

ANALYSIS

Theory of Functions of Complex Variables

Kveselava, D. A. The solution of a boundary problem of the theory of function. C. R. (Doklady) Acad. Sci. URSS (N.S.) 53, 679–682 (1946).

The author solves the following boundary value problem: to find a piecewise analytic function $\varphi(z)$, with a pole of finite order at $z=\infty$, satisfying $\varphi^+(\alpha(t))=G(t)\varphi^-(t)+g(t)$ on a bounded simple closed smooth curve L; G, g are assigned of class H (Hölder) on L ($G\neq 0$ on L); $\alpha(t)$ of class H is assigned on L ($\alpha^{(1)}\neq 0$ on L). The solution is based on the use of singular integral equations with Cauchy kernels studied by N. Mushelishvili and others.

W. J. Trjitzinsky (Urbana, Ill.).

Broggi, Ugo. Sull'ascissa di olomorfia di una funzione. Boll. Un. Mat. Ital. (2) 5, 236–240 (1943).

Quelques considérations élémentaires sur les séries de la forme $\sum a_n\{(s-\alpha)/(s+\beta)\}^r$.

S. Mandelbrojt (Paris).

Robinson, Raphael M. Univalent majorants. Trans. Amer. Math. Soc. 61, 1–35 (1947).

A function $G(z)$, regular and univalent in $|z|<1$, is said to be a majorant of a function $g(z)$, regular in $|z|<1$, if $g(0)=G(0)$ and if the values of g lie in the map of $|z|<1$ by G. For this the symbol $g(z)\prec G(z)$, in $|z|<1$, is used. Similar definitions and notations hold for other circles. The symbol $g(z)\prec\mathfrak{H}\{G(z)\}$ indicates that the values of g lie in the convex cover of the map by G. The main problem of the paper is to investigate when $g(z)\prec G(z)$ in $|z|<1$ implies $\mathfrak{L}g(z)\prec\mathfrak{L}G(z)$ in $|z|<r\leqq 1$. Here $\mathfrak{L}g$ denotes a linear operator of order zero applied to g. The author, however, restricts himself usually to the simple majorants G: $(1+z)/(1-z)$, $1/(1-z)^2$, $\log(1+z)/(1-z)$, and to the operators $\mathfrak{L}g$:

$$zg', \quad [zg]', \quad z^{-1}\int_0^z g(\zeta)d\zeta, \quad \int_0^z \zeta^{-1}\{g(\zeta)-g(0)\}d\zeta.$$

Of the various results the following may be quoted as typical. (i) If $G(z)=(1+z)/(1-z)$ or $1/(1-z)^2$, then $g(z)\prec G(z)$ im-

plies $\mathfrak{L}g(z) \prec \mathfrak{H}\{\mathfrak{L}G(z)\}$ in $|z|<1$, for every operator $\mathfrak{L}$ of order zero. (ii) If $g(z) \prec G(z)$ in $|z|<1$, then $zg'(z) \prec zG'(z)$ in $|z| \leq 3-2^{\frac{1}{2}}$ and $[zg(z)]' \prec [zG(z)]'$ for $|z| \leq \frac{1}{3}$. These radii are best possible. *W. W. Rogosinski.*

Schiffer, Menahem. The kernel function of an orthonormal system. Duke Math. J. 13, 529–540 (1946).

Let $\varphi_\nu(z)$ be a system of analytic functions, orthonormal with respect to a domain B and closed with respect to the class L^2 of functions $f(z)$ analytic in B and with $\iint_B |f(z)|^2 d\omega_z < \infty$ [Bergman]. Let $K(z, \bar{\zeta}) = \sum_{\nu=1}^{\infty} \varphi_\nu(z)\overline{\varphi_\nu(\zeta)}$ be the kernel function of B. The author proves the identity $K(z, \bar{\zeta}) = -(2/\pi)\partial^2 g(z, \zeta)/\partial z \partial \bar{\zeta}$, where $g(z, \zeta)$ is the Green's function of B, $\partial/\partial z = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$, $\partial/\partial \bar{z} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$. The proof is based on the integral equation

$$f(z) = \iint_B K(z, \bar{\zeta}) f(\zeta) d\omega_\zeta.$$

Formulas are given for the variation of $K(z, \bar{\zeta})$. Similar identities are established for orthonormal systems closed with respect to the subclass l^2 of functions possessing a single valued integral in B; these results are related to the conformal representation of B. *J. Ferrand* (Caen).

Fuchs, B. Sur la fonction minimale d'un domaine. II. Rec. Math. [Mat. Sbornik] N.S. 18(60), 329–346 (1946). (Russian. French summary)

[For part I see the same Rec. N.S. 16(58), 21–38 (1945); these Rev. 7, 55.] Let $u = [K_B(t, \bar{t})]^{\frac{1}{2}} \int_t^z M_B(z, t, \bar{t})$, $v = [T_B(t, \bar{t})]^{-\frac{1}{2}}[\partial \log M_B(z, t, \bar{t})/\partial \bar{t}]$, where $K_B(z, \bar{t})$ is the kernel function, $M_B(z, t, \bar{t}) = K_B(z, \bar{t})/K_B(t, \bar{t})$ the minimal function, and $T_B(t, \bar{t}) = \partial^2 \log K_B/\partial t \partial \bar{t}$, map a given domain B into its minimal and representative domains, respectively. (If the domain is simply-connected, the minimal and the representative domains are both circles.) The metrics $A_1: ds^2 = K(z, \bar{z})|dz|^2$ and $A_2: ds^2 = T(z, \bar{z})|dz|^2$ are invariant with respect to conformal transformations. Let C_Γ be the geodesic curvature of certain curves connected with the metric A_2 at the point t and R the Gaussian curvature of the metric A_2. Finally, let S_Γ be the distance (in the metric A_2) between t and z. The author determines α_k such that $(-1)^{k+1}S_\Gamma > \alpha_k|v|$ if $(-1)^{k+1}(C_\Gamma^2 - 4R) > 0$. If $C_\Gamma^2 - 4R > 0$, then there exists a neighborhood of t such that, for every $\beta > \{(C_\Gamma^2 - 4R)/2\}^{\frac{1}{2}}$, $S_\Gamma \leq (2\beta)^{-1} \log \{(1+\beta|v|)/(1-\beta|v|)\}$.

Similar inequalities can be obtained where we substitute for S_Γ the distance in metric A_1 and for v the minimal function of the domain. The above generalizations of the lemma of Schwarz-Pick are of interest because they can be extended to the case of the theory of mapping of four-dimensional domains by pairs of functions of two complex variables and, on the other hand, as has been shown by Bergman [Amer. J. Math. 68, 20–28 (1946); these Rev. 7, 286] and Schiffer [see the preceding review], these functions are connected in a simple manner with functions mapping the domain into a characteristic domain. Further interesting applications can be made if one assumes that t and z approach the boundary. See also Wachs [J. Math. Pures Appl. (9) 22, 25–54 (1943); these Rev. 6, 208]. *S. Bergman* (Cambridge, Mass.).

Heins, Maurice. On the Phragmén-Lindelöf principle. Trans. Amer. Math. Soc. 60, 238–244 (1946).

Der Verfasser kann die Behauptungen des klassischen Phragmén-Lindelöf'schen Prinzips im Falle der Halbebene [R. Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1936; L. Ahlfors, Trans. Amer. Math. Soc. 41, 1–8 (1937)] wesentlich ergänzen. Es sei $f(z)$ in $\Re z > 0$ regulär und $\limsup_{x \to +0} |f(x+iy)| \leq 1$ für alle y, ferner sei $M(r)$ die obere Grenze von $|f(re^{i\theta})|$ für $|\theta| < \pi/2$, $\alpha = \limsup r^{-1} \log M(r)$, $\beta = \liminf r^{-1} \log M(r)$ für $r \to \infty$. Der Verfasser beweist, dass der Fall $-\infty < \alpha < 0$ nie eintritt und dass im Falle $0 < \alpha < \infty$ immer $\alpha = \beta$ ist und zwar entweder $\log M(r) < \alpha r$ für alle $r > 0$ oder $f(z) = Ce^{\alpha z}$ (womit eine von Ahlfors aufgeworfene Frage im positiven Sinne beantwortet ist). Der Beweis stützt sich auf das folgende Lemma. Ist $\phi(z)$ in $\Re z > 0$ regulär und vom Betrage nicht grösser als 1, bezeichnet $E(r, \epsilon)$ das Mass jener θ-Werte im Intervall $|\theta| < \pi/2$, wofür $\log |\phi(re^{i\theta})| < -\epsilon r$ ist und ist $\limsup_{r \to \infty} E(r, \epsilon) > 0$, so gibt es eine Konstante $\kappa > 0$, sodass $\log |\phi(z)| \leq -\kappa \Re z$ in $\Re z > 0$. Der Verfasser gibt ferner einen neuen Beweis für das Phragmén-Lindelöf'sche Prinzip in der Fassung von Ahlfors, nämlich, dass

$$m(r)/r = r^{-1} \int_{-\pi/2}^{\pi/2} \overset{+}{\log} |\phi(re^{i\theta})| \cos \theta d\theta$$

eine nicht abnehmende Funktion von r sei und zeigt, dass $\lim_{r \to \infty} m(r)/r = \frac{1}{2}\pi\alpha$ ist. *A. Pfluger* (Zürich).

Buck, R. Creighton. A class of entire functions. Duke Math. J. 13, 541–559 (1946).

If $f(z)$ is an entire function of exponential type and $h(\theta) = \limsup_{r \to \infty} r^{-1} \log |f(re^{i\theta})|$, $f(z)$ is said to belong to class $K(a, c)$ if $h(0) \leq a$, $h(\pi) \leq a$, $h(\pm\pi/2) \leq c$, and to class K_A if $h(\theta) \leq A$ for all θ. The sequence of complex numbers $\{w_n\}$ being given, a necessary and sufficient condition for the existence of a function $f(z)$ of class $K(a, \pi)$, and even of class K_A, such that $f(n) = w_n$ is that $\limsup_{n \to \infty} |w_n|^{1/n} < \infty$. In addition, using a result of Pólya on Borel transforms [Math. Z. 29, 549–640 (1929), in particular, p. 585], the author gives a necessary and sufficient condition on the sequence $\{w_n\}$ for the existence of a function $f(z)$, belonging to the class $K(a, c)$ for some a and some $c < \pi$, such that $f(n) = w_n$. The condition is that the function $G(z) = \sum_{n=0}^{\infty} w_n z^n$ be regular at the origin and have a continuation which is regular at infinity and in an open set containing the whole negative real axis. A number of applications of this theorem are made; for instance, let $f(z) \epsilon K(a, c)$, $c < \pi$, and $f(n) = r_n e^{i\theta_n}$. Then, if there exists an α, $c < \alpha < \pi$, such that

$$\liminf_{n \to \infty} \{\cos (\theta_n + n\alpha)\}^{1/n} > 0,$$

$f(z)$ vanishes identically. From this follows the corollary that, if D_1 and D_2 are two disjoint closed convex sets in the complex plane and if $w_n = f(n)$ alternates between D_1 and D_2, then $f(z)$ does not belong to any class $K(a, c)$ with $c < \pi$. The author extends these results and discusses the effects of oscillation in sign of the real parts of the numbers $f(n)$. He studies functions of class $K(a, c)$ with $c < \pi$ assuming integral, or "almost integral," values at the integers. The results are applied to show that, p_n denoting the nth prime, there exists a function $f(z)$ of class $K(0, \pi)$ such that $f(n) = p_n$, but no such function of class $K(a, c)$ for $c < \pi$. The results are further extended. *W. Seidel.*

Fuchs, W. H. J. A generalization of Carlson's theorem. J. London Math. Soc. 21, 106–110 (1946).

Let $a_\nu > 0$, $a_{\nu+1} - a_\nu > c > 0$. A necessary and sufficient condition that $f(z)$, regular in $x > 0$ and continuous in $x \geq 0$ and satisfying $f(z) = O(e^{k|z|})$, $k < \pi$, is identically zero if $f(a_\nu) = 0$ is that $(*)$ $\limsup_{r \to \infty} \psi(r) r^{-2k/\pi} = \infty$, where

$$\psi(r) = \exp \left\{ 2 \sum_{a_\nu < r} a_\nu^{-1} \right\}.$$

The interest of the result is that $(*)$ is necessary as well as

sufficient. On the other hand, if $a_\nu/\nu \to D < \pi/k$, (*) is not necessarily sufficient for

$$\limsup_{x\to\infty} x^{-1} \log |f(x)| = \limsup_{\nu\to\infty} a_\nu^{-1} \log |f(a_\nu)|.$$

R. P. Boas, Jr. (Providence, R. I.).

Boas, R. P., Jr. Density theorems for power series and complete sets. Trans. Amer. Math. Soc. 61, 54–68 (1947).

The results are based on the following two uniqueness theorems. (I) Let $F(z)$ be holomorphic in $x>0$, continuous in $x \geqq 0$, $\log |F(z)| < mx \log x + Ax + \sigma(r)$, $m>0$, $\sigma(r)\uparrow$, (1) $\int^\infty t^{-2}\sigma(t)dt < \infty$, and let $F(\lambda_n)=0$, $\lambda_n>0$, with $n_\lambda(t)=n(t) = \sum_{\lambda_n \leqq t} 1 \geqq mt/2 - t\delta(t)$ with $\delta(t)\downarrow 0$, $\int^\infty t^{-1}\delta(t)dt<\infty$. Then $F(z)\equiv 0$. (II) Let $F(z)$ be holomorphic in $x>0$, $\log |F(x+iy)| \leqq k|y| + \sigma(r)$ $(x \geqq 1)$, with (1) satisfied, and let $F(\lambda_n)=0$ with $n(t) \geqq (k/\pi)t + t\delta(t)$, $\delta(t)\downarrow$, $\int^\infty \delta(t)t^{-1}dt = \infty$. Then $F(z)\equiv 0$. Theorem (I) leads to the following results. (A) If $0<\lambda_n\uparrow$ with $n(t)>t/2-t\delta(t)$ with $\delta(t)$ as in (I), then $\{t^{\lambda_n}e^{-t}\}$ is complete with respect to every $L^p(0, \infty)$, $1\leqq p \leqq \infty$. This generalizes a theorem of Fuchs [Proc. Cambridge Philos. Soc. 40, 188–196 (1944); these Rev. 6, 46]. (B) If $a_n>0$, with $n_a(t)>mt/2-t\delta(t)$, m an integer greater than 1, $\delta(t)$ as in (I), then $\{a_n\}$ is a union of m nonoverlapping sequences of which at least one, say $\{\mu_n\}$, is such that $\{t^{\mu_n}e^{-t}\}$ is complete with respect to $L^p(1\leqq p\leqq \infty)$. This generalizes another result of Fuchs in which $m=2$, $\{a_n\}=\{n\}$ [Proc. Cambridge Philos. Soc. 42, 91–105 (1946); these Rev. 7, 294]. A third application of (I) is a uniqueness theorem for the solution of the generalized moment problem. Theorem (II) leads to gap theorems for power series. If $f(z)=\sum c_k z^k$ with $c_k=0$ for $k\neq\lambda_n$ $(0<\lambda_n\uparrow)$ with $n(t)\leqq \pi^{-1}\alpha t - t\delta(t)$ with $\delta(t)$ as in (II), then either $F(z)$ is a constant, or $F(z)$ is either unbounded or has a singular point in every closed angle of opening 2α with vertex at 0. If $\delta(t)\geqq\epsilon>0$ it is a particular case of a theorem of Mandelbrojt [Séries lacunaires, Actualités Sci. Ind., no. 305, Hermann, Paris, 1936]. The author also improves a theorem of Mandelbrojt and Ulrich on quasi-analyticity [same Trans. 52, 265–282 (1942); these Rev. 4, 72]. Some decomposition theorems concerning analytic continuation of gap series and analogous to (B) are also given.

S. Mandelbrojt (Paris).

Boas, R. P., Jr. The growth of analytic functions. Duke Math. J. 13, 471–481 (1946).

Let $\varphi(z)$ be holomorphic for $x\geqq 0$ and let $|\varphi(z)|\leqq Ae^{B|z|}$ there. The author gives a very general solution to the following problem: to give conditions bearing on $\varphi(z)$ and on the sequence λ_n (here $0<\lambda_n\uparrow$) such that the following relationship holds:

$$\limsup_{x\to\infty} x^{-1} \log |\varphi(x)| = \limsup_{n\to\infty} \lambda_n^{-1} \log |\varphi(\lambda_n)|.$$

The theorem contains as a particular case a theorem of V. Bernstein which gives the condition

$$\limsup |y|^{-1} \log |\varphi(iy)| = \pi L,$$

with $L<\lim n/\lambda_n = D$ (this limit is supposed to exist; it is also supposed that $\lambda_{n+1}-\lambda_n\geqq d>0$) [Leçons sur les Progrès Récents de la Théorie des Séries de Dirichlet, Paris, 1933], as well as theorems of Levinson [Gap and Density Theorems, Amer. Math. Soc. Colloquium Publ., v. 26, New York, 1940; these Rev. 2, 180]. The proofs are based on the following theorem, which is interesting in itself: if $f(z)$ is of exponential type in $x\geqq 0$ and is bounded on the curve $\cos\theta=\delta(r)$, $0\leqq\delta(r)<\frac{1}{2}$, $\int^\infty r^{-1}\delta(r)dr=\infty$, then $f(z)$ is bounded in $x\geqq 0$. *S. Mandelbrojt* (Paris).

Akhiezer, N. On some properties of integral transcendent functions of exponential type. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 411–428 (1946). (Russian. English summary)

Let $\omega(z)$ be an entire function of exponential type σ such that $\omega(z)=s(z)+it(z)$, where s and t have no common zeros; $|\omega(z)/\bar\omega(z)|\leqq 1$ for $y\geqq 0$; for every positive ϵ, $|\omega(iy)| \geqq m(\epsilon)e^{(\sigma-\epsilon)|y|}$. If $f(z)$ is an entire function of exponential type $\tau\geqq\sigma$ then $|f(x)|\leqq|\omega(x)|$, $-\infty<x<\infty$, implies (i) that $\frac{1}{2}\{\omega(z)e^{-i(\tau-\sigma)z+i\lambda}+\bar\omega(z)e^{i(\tau-\sigma)z-i\lambda}\}-f(z)$ either is identically zero or has only real zeros, all simple except perhaps those for which $f(x)=\pm\omega(x)$ [generalization of a theorem of Duffin and Schaeffer, Bull. Amer. Math. Soc. 44, 236–240 (1938)]; (ii) that $|f'(x)|\leqq|\{\omega(x)e^{-i(\tau-\sigma)x}\}'|$ and

$$\{f(x)/|\omega(x)|\}' \leqq \tfrac{1}{2}\{\arg[\bar\omega(x)e^{2i(\tau-\sigma)x}/\omega(x)]\}'\{1-[f(x)/|\omega(x)|]^2\}^{\frac{1}{2}}.$$

If $f(x)$ is bounded for real x, the author defines $\tilde f(x)$ by

$$\tilde f(x) = B + (2\pi)^{-\frac{1}{2}}(x-\alpha)\ \text{l.i.m.} \int_a^b e^{ixt} i\varphi(t)\ \text{sgn}\,t\,dt,$$

$$\varphi(t) = (2\pi)^{-\frac{1}{2}}\ \text{l.i.m.} \int_a^b e^{-ixt}(x-\alpha)^{-1}\{f(x)-A\}dx,$$

where B is an arbitrary constant and A and α are such that $(x-\alpha)^{-1}\{f(x)-A\}\epsilon L^2(-\infty,\infty)$; $\tilde f(x)$ is independent of A and α. Then from the interpolation formula

$$f'(x)\sin\theta+\tilde f'(x)\cos\theta = \tau\sum_{k=-\infty}^{\infty}(-1)^{k-1}\frac{2\sin^2\frac{1}{2}(\theta-k\pi)}{(\theta-k\pi)^2}f\{x+\tau^{-1}(k\pi-\theta)\}$$

he derives the inequality

$$|f'(x)\sin\theta+\tilde f'(x)\cos\theta|\leqq\tau\sup|f(x)|,$$

which generalizes results of Szegö [Schr. Königsberg. Gel. Ges. Naturwiss. Kl. 5, 59–70 (1928)] for trigonometric polynomials and Boas [Trans. Amer. Math. Soc. 40, 287–308 (1936)] for trigonometric integrals. *R. P. Boas, Jr.*

Littlewood, J. E., and Offord, A. C. On the distribution of the zeros and a-values of a random integral function. I. J. London Math. Soc. 20, 130–136 (1945).

Die Verfasser teilen ohne Beweis mit, dass bei gegebener Folge $a_1, \cdots, a_n, \cdots$ "fast alle" ganzen Funktionen der Form $f(z)=\sum\pm a_n z^n$ ein regelmässiges asymptotisches Verhalten aufweisen hinsichtlich Betrag und Wertverteilung. Insbesondere ist jede Richtung eine "Borel'sche Richtung," wodurch ein Resultat von Pólya [Math. Z. 29, 549–640 (1929)] über ganze Funktionen unendlicher Ordnung verallgemeinert und verschärft wird. Ferner sei $n(z, r, a)$ die Anzahl der a-Stellen in $|\zeta - z|<r$ und

$$\gamma(z,r,a)=\int_{R^{-p-1}}^{r}u^{-1}n(z,u,a)du, \qquad |z|=R.$$

Anderseits sei $m(R)=\max_n\{|a_n|R^n\}$, $N(R)$ der Index des Maximalterms, $m(R,R')=|a_{N(R)}|R'^{N(R)}$ und

$$I(R,r)=(2\pi)^{-1}\int_{-\pi}^{\pi}\log\frac{m(|R+re^{i\theta}|)d\theta}{m(R,|R+re^{i\theta}|)}.$$

Dann gilt für "fast alle" Funktionen der Form $f(z)=\sum\pm a_n z^n$ und $|z|>R_0$

$$\gamma(z,r,a)=I(R,r)+O(R^\epsilon), \qquad \epsilon>0,$$

gleichmässig in r und a $(|a|<m(R))$. *A. Pfluger.*

Bernstein, S. Sur la meilleure approximation sur tout l'axe réel par des fonctions entières de degré fini. IV. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 103–108 (1946).

Starting with an interpolation formula [similar to a result given in part III, same C. R. (N.S.) **52**, 563–566 (1946); these Rev. **8**, 323], the author proves the following results. (1) If (i) $F(z)$ is an entire function of exponential type p, (ii) $F(x)=o(x)$, (iii) $F(k\pi/p)=O(|k|^\alpha)$ $(k=0, \pm1, \pm2, \cdots; 0\leqq\alpha<1)$, then there are trigonometric sums

$$S_{n,p}(x)=\sum_{k=0}^{n}(a_k\cos kpx/n+b_k\sin kpx/n)$$

such that $|F(x)-S_{n,p}(x)|\leqq M|x|n^{\alpha-1}$. (2) If $F(x)$ is of type (i) and $|F(x)|\leqq\max(N, M|x|^\alpha)$ $(0\leqq\alpha<\infty)$, then there are sums $S_{n,p}$, which are constructed, such that $|F(x)-S_{n,p}(x)|\to0$ uniformly in $(-L_n, L_n)$, where $L_n\to\infty$ $(n\to\infty)$, L_n depending on α and n.

Then the best approximations of a function $f(x)$ by sums $S_{n,p}$ and by polynomials, both majorized by certain even monotonic functions, on $(-L, L)$ are compared with the best approximation by an $F(x)$ of type (i) on $(-\infty, \infty)$. Thus a result of the previous paper is generalised and a corollary deduced: if a bounded function $f(x)$ is uniformly continuous in $(-\infty, \infty)$, then, given any $\epsilon>0$ and $L>0$, there are numbers n, p $(p\leqq P; P=P(\epsilon)$ independent of $L)$ such that $|f(x)-S_{n,p}(x)|\leqq\epsilon$ on $(-L, L)$; the condition is necessary. *H. Kober* (Birmingham).

Cowling, V. F. A generalization of a theorem of LeRoy and Lindelöf. Bull. Amer. Math. Soc. **52**, 1065–1082 (1946).

The author characterizes the behavior of $f(z)$ given by $\sum a_nz^n$ with $a_n=a(n)$, $a(z)$ being holomorphic in an angular opening including the real positive axis; this is done in terms of the angles ψ_1 and ψ_2 which the opening (in which $a(z)$ is holomorphic) makes with the real axis. The behavior of $a(z)$ at infinity plays, of course, an important role. The results constitute generalizations of well-known theorems of LeRoy and Lindelöf. [The reviewer believes that it should be noticed that deep theorems in this direction were proved by V. Bernstein in various papers.] *S. Mandelbrojt.*

Nehari, Zeev. On certain classes of meromorphic functions. J. London Math. Soc. **20**, 219–225 (1945).

(1) Bezeichnet a_ν die Nullstellen, β_ν die 1-Stellen und γ_ν die einfachen Pole der in $|z|<\infty$ meromorphen Funktion $f(z)$ und sind die $\alpha_\nu, \beta_\nu, \gamma_\nu$ ganze rationale Zahlen, so gilt $|f'(\alpha_\nu)|\leqq2\pi$, $|f'(\beta_\nu)|\leqq2\pi$, $\lim_{z\to\gamma_\nu}|(z-\gamma_\nu)f(z)|\geqq(2\pi)^{-1}$. Diese Schranken sind die bestmöglichen. Auch für $f(\alpha_\nu\pm t)$, $f(\beta_\nu\pm t)$, $f(\gamma_\nu\pm t)$ werden Schranken angegeben für reelles t, $0\leqq t<1$. (2) Sind die $\alpha_\nu, \beta_\nu, \gamma_\nu$ ganze Gauss'sche Zahlen $m+in$ $(m, n=0, \pm1, \pm2, \cdots)$, so gilt $|f'(\alpha_\nu)|\leqq(2/\pi)^{\frac{1}{2}}\Gamma^2(\frac{1}{4})$, $|f'(\beta_\nu)|\leqq(2/\pi)^{\frac{1}{2}}\Gamma^2(\frac{1}{4})$, $\lim_{z\to\gamma_\nu}|(z-\gamma_\nu)f(z)|\geqq\{(2/\pi)^{\frac{1}{2}}\Gamma^2(\frac{1}{4})\}^{-1}$. Diese Schranken sind die bestmöglichen. Der Beweis gelingt mit Hilfe des "principle of subordination" durch Heranziehen geeigneter Transzendenten, des trigonometrischen und des Jacobischen Sinus und der elliptischen Modulfunktion. *A. Pfluger* (Zürich).

Onofri, Luigi. Sugli zeri della derivata di una funzione quasi intera con due punti singolari. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **12**, 311–334 (1942).

Der Verfasser verallgemeinert die Untersuchungen von Laguerre und andern über die Nullstellen der Ableitung gewisser ganzer Funktionen auf sogenannte quasiganze Funktionen, d.h., solche, die in $0<|z|<\infty$ eindeutig und analytisch sind. Die quasiganze Funktion $f(z)$ sei vom Geschlecht p, sei reellwertig auf der reellen Achse und habe q nichtreelle Nullstellen. Es bezeichne I_j die Intervalle zwischen zwei konsekutiven Nullstellen gleichen Vorzeichens für $f(z)$ (von der Länge null bei mehrfachen Nullstellen), r_j die Nullstellenzahl der Ableitung im Intervall I_j, s deren Anzahl ausserhalb aller Intervalle I_j. Dann gilt $\sum(r_j-1)+s\leqq2p+q+2$. Der Beweis stützt sich auf eine zur Weierstrass'schen analogen kanonischen Produktdarstellung quasiganzer Funktionen [E. Maillet, J. Math. Pures Appl. (5) **8**, 19–57 (1902)]. Verschiedene Spezialfälle werden genauer untersucht. *A. Pfluger* (Zürich).

Tsuji, Masatsugu. On the behaviour of a meromorphic function in the neighbourhood of a closed set of capacity zero. Proc. Imp. Acad. Tokyo **18**, 213–219 (1942). [MF 14754]

The program of this note is closely related to the work of G. af Hällström [Acta Acad. Aboensis **12**, no. 8 (1940); these Rev. **2**, 275]. *M. Heins* (Providence, R. I.).

Tsuji, Masatsugu. On non-prolongable Riemann surfaces. Proc. Imp. Acad. Tokyo **19**, 429–430 (1943). [MF 14838]

Remarks on Radó's example of a noncontinuable Riemann surface [Math. Z. **20**, 1–6 (1924)]. *M. Heins.*

***Martinelli, E. Intorno alla teoria delle funzioni biarmoniche e delle funzioni analitiche di due variabili complesse.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 162–167. Edizioni Cremonense, Rome, 1942.

Announcement of results. Cf. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **12**, 143–167 (1942); these Rev. **8**, 203.

Ridder, J. Über areolär-monogene Funktionen. Nieuw Arch. Wiskunde (2) **22**, 200–206 (1946).

Let C be a simple closed rectifiable Jordan curve bounding a finite region B in the plane. If $u(z)=u_1(x, y)+iu_2(x, y)$, $z=x+iy$, is continuous in $B+C$, then $\Phi_u(J)\equiv\int_R u(z)dz$, where J is a closed oriented square lying in B and R is its boundary, is a function of squares associated with $u(z)$. Upper and lower derivates are defined by the relations

$$D_z^+\phi_u=\lim_{\|J\|\to0}\sup\phi_u(J)/\|J\|,\quad D_z^-\phi_u=\lim_{\|J\|\to0}\inf\phi_u(J)/\|J\|,$$

where $z\epsilon J$. If $D_z^+\phi_u$ and $D_z^-\phi_u$ are finite and if $D_z^+\phi_u=D_z^-\phi_u$, then their common value is denoted by $D_z\phi_u$ and $u(z)$ is said to be areolar-monogenic at z. If $u(z)$ is continuous in $B+C$, then the author shows that $D_z^+|\phi_u|$ can be infinite in at most a countable set in B. If in addition $D_z^+\phi_{u_1}$ and $D_z^+\phi_{u_2}$ are Lebesgue integrable, then $u(z)$ is areolar-monogenic almost everywhere in B and $\int_C u(z)dz=\iint_B D_z\phi_u dxdy$. Moreover, if $z_0\epsilon B$, then

$$2\pi iu(z_0)=\int_C(z-z_0)^{-1}u(z)dz-\iint_B(z-z_0)^{-1}D_z\phi_u dxdy,$$

which is a well-known result [see Théodoresco, La dérivée aréolaire et ses applications à la physique mathématique, thèse, Paris, 1931]. The present proof of this last result is simpler than the earlier ones and is more general. Finally, if $z_0\bar{\epsilon}B+C$, then

$$0=\int_C(z-z_0)^{-1}u(z)dz-\iint_B(z-z_0)^{-1}D_z\phi_u dxdy.$$

M. O. Reade (Ann Arbor, Mich.).

Ridder, J. Einige einfache Anwendungen der areolären Ableitungen und -Derivierten. Nederl. Akad. Wetensch., Proc. **50**, 151–156 = Indagationes Math. **9**, 114–119 (1947).

The author applies known theorems in the theory of linear transformations to some of his more recent results involving areolar derivatives [Acta Math. **78**, 205–289 (1946); these Rev. **8**, 271] to obtain additional results of the following type. Let B denote a bounded Dirichlet region in the plane and let C denote its boundary. Three classes of complex-valued functions $f(x, y) = u(x, y) + iv(x, y)$, which are defined and measurable in $B+C$, are defined as follows: L_2 is the class of functions of integrable square on B; R is the largest subset of L_2 containing only bounded functions, where two functions that differ on at most a set of measure zero are said to be identical; D is the largest subset of L_2 for which $f(x, y) = 0$ on C and u and v have bounded upper and lower areolar derivatives. The areolar derivatives $D_{(x,y)}\Phi_u$ and $D_{(x,y)}\Phi_v$ of $u(x, y)$ and $v(x, y)$, respectively, exist almost everywhere for $f = (u+iv)\epsilon D$. It can then be shown that the linear operator T defined by

$$Tf = D_{(x,y)}\Phi_u + iD_{(x,y)}\Phi_v = \varphi(x, y) + i\psi(x, y)$$

defines a one-to-one map of D on R. The inverse T^{-1} is defined in R and is an integral operator of Hilbert-Schmidt type with symmetric kernel:

$$u+iv = T^{-1}(\varphi+i\psi) = (2\pi)^{-1}\iint_B g(\xi, \eta; x, y)\{\varphi(\xi, \eta) + i\psi(\xi, \eta)\}d\xi d\eta,$$

where $g(\xi, \eta; x, y)$ is Green's function for B for the point (x, y). The operators T and T^{-1} are essentially self adjoint in the usual Hilbert space arising from L_2. Moreover, there is a denumerable infinity of eigenvalues for T, $\{\lambda_n\}$, such that λ_n is real, $\lambda_n \neq 0$, and such that the λ_n are isolated. For each λ_n there exists a finite number of mutually orthogonal eigenfunctions, such that the totality of these eigenfunctions constitutes a complete system in the usual Hilbert space associated with L_2. Furthermore, if λ is complex, $\lambda \neq \lambda_n$, $n = 1, 2, \cdots$, and if $g\epsilon L_2$, then there is at most one $f\epsilon D$ such that $Tf - \lambda f = g$. Analogous results hold for functions of three variables and for functions having areolar derivatives of order greater than one. *M. O. Reade.*

Theory of Series

Popoviciu, Tiberiu. On monotone series. Pozitiva **1**, 41–45 (1940). (Romanian. French summary)

It is first shown that every sequence of $\frac{1}{2}n!$ elements ($n \geqq 3$) contains a subsequence of n elements which is monotone (nondecreasing or nonincreasing). There therefore exists a number N_n such that every sequence of N_n elements contains a monotone subsequence of n elements, while there is a sequence of $N_n - 1$ elements having no monotone subsequence of n elements. The first statement implies that $N_n \leqq \frac{1}{2}n!$. On the other hand the following special sequence of $(n-1)^2$ elements, $n-1, n-2, \cdots, 2, 1, 2n-2, 2n-3, \cdots, n, 3n-3, 3n-4, \cdots, 2n-1, \cdots, (n-1)^2, (n-1)^2-1, \cdots, (n-1)^2-(n-2)$, has no monotone subsequence of n elements. Hence $(n-1)^2 < N_n \leqq \frac{1}{2}n!$. The author adds, without proof, that very likely $N_n = (n-1)^2+1$, for all values of n. [This is actually true, as was shown in a somewhat wider context by P. Erdös and G. Szekeres, Compositio Math. **2**, 463–470 (1935).] *I. J. Schoenberg* (Philadelphia, Pa.).

Pettineo, B. Interdipendenza tra serie convergenti e serie divergenti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 934–937 (1946).

Let $U_1, U_2, U_3, \cdots$ be an increasing sequence of positive numbers. If one of the two series

$$\sum u_n = U_1 + (U_2 - U_1) + (U_3 - U_2) + \cdots,$$
$$\sum a_n = (U_1^{-1} - U_2^{-1}) + (U_2^{-1} - U_3^{-1}) + (U_3^{-1} - U_4^{-1}) + \cdots$$

is divergent, then the other is convergent. By use of this "elementare principio," propositions about convergent (or divergent) series are converted into propositions about divergent (or convergent) series. The propositions featured are propositions of Abel and Dini (concerning $\sum u_n/U_n$ and $\sum a_n/U_{n-1}^{-1}$) and generalizations of them. *R. P. Agnew.*

Levi, F. W. Rearrangement of convergent series. Duke Math. J. **13**, 579–585 (1946).

The author considers permutations of the terms of real series which do not alter their sums; S^* denotes the set of permutations which leave the sum of the series S invariant, V_0 the set which has this property with respect to any convergent series. A simple necessary and sufficient condition is given for a permutation Π to belong to V_0. Furthermore, it is proved that V_0 is not a group, but a semigroup, and that, for any conditionally convergent series S, a permutation $\alpha\epsilon V$ exists such that $(\alpha S)^*$ is not even a semigroup. However, the problem of whether S^* itself can be a semigroup remains unsolved.

From the binary relation $S\rho\Pi$ (Π leaves the sum of S invariant) a notion of closure of sets of series and of permutations is derived. With respect to series of complex numbers and of vectors, it is remarked that these give the same lattice of closed sets of permutations as the real series do. *N. G. de Bruijn* (Delft).

Goodstein, R. L. A theorem in uniform convergence. Math. Gaz. **30**, 287–290 (1946).

This paper gives three proofs of the known theorem that, if $\varphi_n(x)$ is continuous, approaches zero in $a \leqq x \leqq b$ and $\varphi_{n+1}(x) < \varphi_n(x)$, then $\varphi_n(x)$ approaches zero uniformly in $a \leqq x \leqq b$. *T. Fort* (Athens, Ga.).

Rogers, C. A. Linear transformations which apply to all convergent sequences and series. J. London Math. Soc. **21**, 123–128 (1946).

Rogers, C. A. Addendum to "Linear transformations which apply to all convergent sequences and series." J. London Math. Soc. **21**, 182–185 (1946).

Let the transformation $A = \|a_k(\omega)\|$ ($k = 1, 2, \cdots; \omega \to \infty$) be said to apply to the sequence s_k (or to the series $\sum c_k$) provided that for all sufficiently large values of the parameter ω the series $\sum a_k(\omega)s_k$ (or the series $\sum a_k(\omega)c_k$) converges. The author proves the following. If the transformation A applies to every null sequence, there exists a constant ω_0 such that $\sum|a_k(\omega)| < \infty$ when $\omega > \omega_0$; it follows then that for every convergent sequence s_k the series $\sum a_k(\omega)s_k$ converges absolutely when $\omega > \omega_0$. If the transformation A applies to every convergent series, there exists a constant ω_0 such that $\sum|a_k(\omega) - a_{k+1}(\omega)| < \infty$ when $\omega > \omega_0$; it follows then that for every convergent series $\sum c_k$ the series $\sum a_k(\omega)c_k$ converges when $\omega > \omega_0$.

In the addendum the author points out that the results can be obtained more easily from a theorem of S. Banach. In a note appended to the addendum he reports that the first of his theorems was proved by R. P. Agnew [Bull.

Amer. Math. Soc. **45**, 689–730 (1939), p. 709; these Rev. **1**, 50]. *G. Piranian* (Ann Arbor, Mich.).

Good, I. J. On the regularity of a general method of summation. J. London Math. Soc. **21**, 110–118 (1946).

If for each $n=0, 1, 2, \cdots$ the series in brackets converges to a number different from zero, then the series

$$(1) \qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left\{\varphi(n,k)u_k\Big/\left[\sum_{j=0}^{\infty}\varphi(n,j)u_j\right]\right\}a_n$$

obviously converges to s when and only when $\sum a_n$ does. After imposing six conditions on $\varphi(n,k)$ and u_k, the author starts with the formula

$$(2) \qquad s=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left\{\varphi(n,k)u_k\Big/\left[\sum_{j=0}^{\infty}\varphi(n,j)u_j\right]\right\}a_n$$

in which the order of summation over k and n is opposite to that in (1). The series $\sum a_n$ is called summable $\{\varphi(n,k), u_k\}$ to s if (2) holds. The six conditions imply that $\{\varphi(n,k), u_k\}$ is regular, that is, that (2) holds whenever $\sum a_n$ converges to s. The proof consists in denoting the sum over $k=1, 2, \cdots, N$ of terms in (2) by t_N, changing order of summation, writing the result in terms of the partial sums of $\sum a_n$, and applying the Silverman-Toeplitz conditions for regularity. Among the standard methods of summability included among these methods are the Cesàro methods C_r, the Euler methods E_r and the Weierstrass circle-method. [Reviewer's remark. The latter method, sometimes called the "Hardy-Littlewood circle-method," is precisely the familiar "Weierstrass circle-method" of extending values of a function of a complex variable from the circle of convergence of a power series to regions where the series diverges.] *R. P. Agnew* (Ithaca, N. Y.).

Barlaz, Joshua. On some triangular summability methods. Amer. J. Math. **69**, 139–152 (1947).

The Borel transformation replaces an arbitrary sequence s_n by the function $e^{-x}\sum_{k=0}^{\infty}s_kx^k/k!$. The present paper considers the related row-finite Toeplitz transformation (1) $t_n=e^{-x_n}\sum_{k=0}^{n}s_kx_n^k/k!$, where x_n is an arbitrary (but fixed) sequence of real numbers. In all that follows, it is assumed that $\lim x_n=\infty$. For every sequence x_n the transform (1) of the sequence $1, 0, 1, 0, \cdots$ converges, but there exists a sequence s_n, summable by the method of arithmetic means, whose transform (1) does not converge. A necessary and sufficient condition for convergence of s_n to imply convergence of t_n is the existence of the quantity $\rho=-\lim (x_n-n)/n^{\frac{1}{2}}$; if $s=\lim s_n$ and ρ exists,

$$\lim t_n=s(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\rho}\exp(-t^2/2)dt.$$

If $\lim x_n/n=0$ and the power series $\sum a_nz^n$ has a positive radius of convergence, the transformation (1) sums the sequence $s_n(z)=\sum_{k=0}^{n}a_kz^k$ whenever the Borel method sums it, and to the same value. If $x_n\leqq n/e+K$, where K is constant, the transformation (1) sums every Fourier series whenever the Borel method sums it, and to the same value. *G. Piranian* (Ann Arbor, Mich.).

Kuttner, B. Note on strong summability. J. London Math. Soc. **21**, 118–122 (1946).

A series $\sum u_n$ and its sequence s_n of partial sums are strongly summable of order q (summable H_q) to s if

$$\lim_{n\to\infty} n^{-1}\{|s_1-s|^q+|s_2-s|^q+\cdots+|s_n-s|^q\}=0.$$

This method H_q for evaluating series has been applied to Fourier series by several authors. The present paper compares H_q with Silverman-Toeplitz methods. If $q>1$, each series summable H_q is summable by the Cesàro method C_r when $r>1/q$ but not necessarily when $r=1/q$. Each series summable H_1 is summable C_1, but not necessarily summable C_r when $r<1$. When $q<1$, the results are different. The main theorem is the following. If $q<1$ and T is a regular Silverman-Toeplitz method, then there is a series which is summable H_q but nonsummable T. Finally, the author gives an example of a regular T, which is stronger than convergence, such that each series summable T is summable H_q for each $q>0$. *R. P. Agnew* (Ithaca, N. Y.).

Rajagopal, C. T. On converse theorems of summability. Math. Gaz. **30**, 272–276 (1946).

Let $0=\lambda_0<\lambda_1<\cdots$ and let $\lambda_n\to\infty$. A series $\sum a_n$ with partial sums s_n is summable to s by the Riesz method $R(\lambda_n, 1)$ of order 1 if the first of the limits

$$\lim_{n\to\infty}\lambda_{n+1}^{-1}\sum_{k=1}^{n}(\lambda_{k+1}-\lambda_k)s_k,\quad \lim_{t\to 0+}\sum_{n=1}^{\infty}a_ne^{-\lambda_nt}$$

is s, and is summable to s by the Dirichlet series method $D(\lambda_n)$ if the second limit is s. Short proofs are given for several known Tauberian theorems for these methods of summability. *R. P. Agnew* (Ithaca, N. Y.).

Wintner, Aurel. On the Tauberian nature of Ikehara's theorem. Amer. J. Math. **69**, 99–103 (1947).

Ikehara's theorem for ordinary Dirichlet series may be stated as follows. If (i) $f(s)=\sum a_nn^{-s}$ $(s=\sigma+it)$, where the series is absolutely convergent in the half-plane $\sigma>1$ and the function $f(s)$ is continuous for $\sigma\geqq 1$, (ii) $a_n=O_L(1)$, then $\sum_1^xa_n=o(x)$. It is shown by an example that the hypothesis (ii) cannot be omitted. This is contrasted with the situation that arises with the Tauberian theorem for Lambert summability. If (i′) $\sum c_n$ is (L)-summable, (ii′) $c_n=O_L(1/n)$ [the author's (2 bis) is presumably a slip for this], then $\sum c_n$ is convergent. Here it is known that we may omit the Tauberian condition (ii′) at the expense of replacing the conclusion by "$\sum c_n$ is (A)-summable"; so that we have a pure Abelian inference from (L)- to (A)-summability. It is apparently inferred from the example cited above that Ikehara's theorem cannot be deprived of its Tauberian character in a similar manner. [The comparison seems, however, imperfect, in that there is no proposal to balance the omission of the hypothesis (ii) by a corresponding weakening of the conclusion.] *A. E. Ingham.*

Hilding, S. H. On infinite sets of homogeneous linear equations in Hilbert space. Quart. J. Math., Oxford Ser. **17**, 240–244 (1946).

The following theorem is obtained. The infinite set of equations $\sum_{q=1}^{\infty}x_q/(\lambda_q-\mu_q)=0$, $p=1, 2, \cdots$, where $\lambda_p\neq\mu_q$ for all p, q; $\mu_p\neq\mu_q$ for $p\neq q$, has no solution $\{x_q\}$ in l^2 except $\{0\}$, provided that, for all $p\geqq 1$, $\lim_{N\to\infty}\sum_{q=N+1}^{\infty}|R_q(N)|^2<\infty$; $R_q(N)$ is defined as the product

$$\prod_{i=1,\, i\neq p}^{N}(\mu_i-\mu_q)(\mu_i-\mu_p)^{-1}\prod_{i=1}^{N}(\lambda_i-\mu_i)(\lambda_i-\mu_q)^{-1}.$$

H. Pollard (Ithaca, N. Y.).

Fourier Series and Generalizations, Integral Transforms

Turetzky, A. Asymptotical inequalities for trigonometrical polynomials satisfying a differential relation at a certain system of points. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 487–512 (1946). (Russian. English summary)

L'auteur donne la solution du problème suivant: soit $x_k=2kn/(2n+1)$ $(0\leqq k\leqq 2n)$; les quantités réelles $a_0, \cdots, a_m$ étant données, détermine la valeur asymptotique de la borne supérieure du module des polynomes trigonométriques $P_n(x)$ de degré m pour lesquels $\sum_{\nu=0}^{m}|a_\nu P_n^{(\nu)}(x_k)|\leqq 1$, $0\leqq k\leqq 2n$.
S. Mandelbrojt (Paris).

***Foà, Alberto. Sulla sommabilità assoluta $|C, \alpha|$ delle serie di Fourier di una funzione sommabile L^p con $p>1$.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 152–153. Edizioni Cremonense, Rome, 1942.

Announcement of a theorem subsequently published in Boll. Un. Mat. Ital. (2) 2, 325–332 (1940); 3, 393–394 (1941); these Rev. 2, 94; 3, 105.

Hardy, G. H., and Rogosinski, W. W. Notes on Fourier series. IV. Summability (R_2). Proc. Cambridge Philos. Soc. 43, 10–25 (1947).

[For note III see Quart. J. Math., Oxford Ser. 16, 49–58 (1946); these Rev. 7, 247.] The paper is devoted to the study of a summation method for Fourier series. The method (R_2) is defined by

$$\sum a_n=s(R_2)\leftrightarrow\lim_{h\to 0} 2\pi^{-1}h^{-1}\sum s_n(n^{-1}\sin nh)^2=s.$$

It is closely related to a better-known summation method $(R, 2)$ defined by $\lim_{h\to 0}\sum a_n(n^{-1}h^{-1}\sin nh)^2=s$. Suppose that E is the set of points such that for every $x\epsilon E$ there exists a c for which $\lim_{t\to 0}t^{-1}\int_0^t\{f(x+u)+f(x-u)-2c\}du=0$. It is proved that the Fourier series of f is (R_2) summable to c at all points of E, and hence almost everywhere. Two necessary and sufficient conditions are given for the (R_2) summability of the Fourier series of f at a point x. (i) There exists a c such that

$$\int_{0+}^{\pi}\log\left|\frac{t+h}{t-h}\right|\frac{dt}{t^2}\int_0^t\{f(x+u)+f(x-u)-2c\}du$$

converges as a Cauchy integral down to 0 and tends to 0 with h. (ii) Denoting by $\bar{f}$ the function conjugate to f, supposed to be integrable, then

$$\pi^{-1}\int_{0+}^{\infty}t^{-2}dt\int_0^t\{\bar{f}(x+u)-\bar{f}(x-u)\}du$$

is convergent. Finally two examples of Fourier series are constructed of which one is summable $(R, 2)$, but not (R_2), and the other (R_2), but not $(R, 2)$. *František Wolf.*

Minakshisundaram, S., and Szász, Otto. On absolute convergence of multiple Fourier series. Trans. Amer. Math. Soc. 61, 36–53 (1947).

The authors prove that, if

$$\text{(1)}\qquad f(x_1, \cdots, x_k)\sim\sum c_{n_1,\cdots,n_k}e^{i(n_1x_1+\cdots+n_kx_k)}$$

satisfies the Lipschitz condition (2) $f(P)-f(Q)=O(|P-Q|^\alpha)$, then $\sum|c_{n_1,\cdots,n_k}|^\beta<\infty$ for $\beta>2k/(k+2\alpha)$, and this is a best result. It should be noted that, for $k\geqq 2$, $\beta=1$ is not compatible with $0<\alpha\leqq 1$, and thus no genuine criterion for absolute convergence of (1) is implied. The authors could have concluded that, if $f(x)$ has partial derivatives of order less than or equal to g, and if the latter satisfy the Lipschitz condition, then absolute convergence is obtained for $g>\frac{1}{2}k-\alpha$. The authors prove their theorem by extending a previous proof of Szász from $k=1$ to $k\geqq 2$ and they employ the technique of spherical summability which was introduced by the reviewer. This is noteworthy since "sphericality" does not explicitly enter into the assertion. The bulk of the paper is made up of a generalization. The "modulus of continuity" $\omega(t)=|t|^\alpha$ in the assumption (2) is replaced by a general function $\omega(t)$, but the general function then enters into the assertion. A further generalization is obtained by setting up the assumption (2), or its generalizations, not for the function $f(x)$ itself, but for its spherical means around arbitrary points. *S. Bochner.*

Chelidze, V. G. On the absolute convergence of double Fourier series. C. R. (Doklady) Acad. Sci. URSS (N.S.) 54, 117–120 (1946).

Let $f(x, y)$ be a periodic function of the two variables x, y. Under the conditions

$$|f(x_1, y)-f(x_2, y)|\leqq H(y)|x_1-x_2|^\alpha,$$
$$|f(x, y_1)-f(x, y_2)|\leqq K(x)|y_1-y_2|^\beta,$$
$$|f(x_1, y_1)-f(x_2, y_1)+f(x_2, y_2)-f(x_1, y_2)|\leqq C|x_1-x_2|^\gamma|y_1-y_2|^\delta,$$

with $H(y)\epsilon L$, $K(x)\epsilon L$, C constant, and α, β, γ, δ all greater than $\frac{1}{2}$, the author proves the absolute convergence of the double Fourier series of f, thus extending to two variables the well-known theorem of S. Bernstein. *R. Salem.*

Cesari, L. Sulle serie doppie di Fourier. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 1173–1175 (1946).

The author observes that a recent result of Grünwald [Acta Litt. Sci. Szeged 10, 105–108 (1941); these Rev. 7, 293] concerning the summability of double Fourier series can be restated as follows: the sequence of the "triangular" partial sums of the Fourier series of a function $f(x, y)$ is summable $(C, 1)$ to $f(x, y)$ at almost every point.
A. Zygmund (Chicago, Ill.).

Amerio, Luigi. Sulla convergenza in media della serie $\sum_0^\infty {}_n a_n e^{i\lambda_n x}$. Ann. Scuola Norm. Super. Pisa (2) 10, 191–198 (1941).

The author considers generalized trigonometric series of the form

$$\text{(1)}\qquad \sum_{n=0}^{\infty}a_ne^{i\lambda_nx},\qquad \lambda_{n+1}-\lambda_n\geqq\delta>0.$$

He defines uniform convergence in mean over the interval $(-\infty, \infty)$ to mean

$$\lim_{N\to\infty}\int_k^{k+1}\left|\sum_{n=0}^{N}a_ne^{i\lambda_nx}-F(x)\right|^2dx=0,$$

uniformly in k, $-\infty<k<\infty$. The principal result of the paper is that, if the two series

$$\sum_{n=0}^{\infty}|a_n|^2,\quad \sum_{n=0}^{\infty}\frac{|a_n|^2+|a_{n+1}|^2}{\lambda_{n+1}-\lambda_n}$$

converge, the series (1) converges uniformly in mean over the interval $(-\infty, \infty)$. *R. Bellman* (Princeton, N. J.).

Bohr, Harald. On S-almost-periodic functions with linearly independent exponents. Norsk Mat. Tidsskr. **26**, 33–40 (1944). (Danish)

It is known that, if $\sum a_{\lambda_n}e^{i\lambda_n x}$ has linearly independent exponents and corresponds to a continuous almost periodic function, then $\sum|a_{\lambda_n}|<\infty$. The author shows that continuity is necessary and that the same result is not true for almost periodic functions in the sense of Stepanoff. In particular, he shows that, to an arbitrary sequence of complex numbers a_n such that $\sum|a_n|^2<\infty$, we can find a sequence of linearly independent exponents $\{\lambda_n\}$ so that there exists an f such that

$$\lim_{N\to\infty}\max_{x}\int_x^{x+1}\left|f(y)-\sum_1^N a_{\lambda_n}e^{i\lambda_n y}\right|^2dy=0.$$

Such a function is known to be S_2-almost periodic. But the following result also holds. If $\sum a_{\lambda_n}e^{i\lambda_n x}$ has its linearly independent exponents bounded and corresponds to an S-almost periodic function, then $\sum|a_{\lambda_n}|<\infty$. Boundedness of the exponents of an S-almost periodic function makes it an analytic almost periodic function and so this case is easily reduced to the classical one. *František Wolf.*

Cinquini, Silvio. Sopra i polinomi di Bochner-Féjer e le funzioni quasi-periodiche secondo Stepanoff. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **9(78)**, 391–400 (1945).

Soit $f(x)$ définie presque-partout pour $-\infty<x<\infty$, sommable sur tout intervalle fini et développable en série de Fourier généralisée (c'est-à-dire que la valeur moyenne de $f(x)e^{-i\lambda x}$ existe pour toutes les valeurs réelles de λ, mais n'est différente de zéro que pour une infinité dénombrable de valeurs de λ); soit $\phi(u)$ une fonction définie pour $u\geqq 0$, non négative et convexe; alors si $\phi(|f(x)|)$ est sommable sur tout intervalle fini et de plus uniformément sommable (ce qui veut dire qu'à tout $\epsilon>0$ on peut faire correspondre un $\delta>0$ tel que sur tout ensemble de mesure non supérieure à δ contenu dans un intervalle de longueur 1, l'intégrale de $\phi(|f|)$ sur cet ensemble ne dépasse pas ϵ), alors les intégrales $\int\phi(|\sigma_{B_p}(x)|)dx$, $p=1, 2, \cdots$, où σ_{B_p} est le polynome de Bochner-Fejér d'ordre p de $f(x)$, sont équiabsolument continus de manière uniforme. Si, de plus, $\phi(0)=0$, si $f(x)$ est presque-périodique S et si $\phi(2|f|)$ est sommable sur tout intervalle fini et de plus uniformément sommable, alors

$$\lim_{p\to\infty}\limsup_{-\infty<a<+\infty}\int_a^{a+1}\phi(|f(x)-\sigma_{B_p}(x)|)dx=0.$$

J. Favard (Paris).

McLachlan, N. W. A general theorem in Laplace transforms. Math. Gaz. **30**, 85–87 (1946).

Define $f(t)\supset\varphi(p)$ to mean that $\varphi(p)=p\int_0^\infty e^{-pt}f(t)dt$. It is proved that under suitable conditions the hypotheses $f(t)\supset\varphi(p)$ and $\xi(x, t)\supset\varphi_1(p)h(p)e^{-xh(p)}$ imply that $\int_0^\infty\xi(x, t)f(x)dx\supset\varphi_1(p)\varphi[h(p)]$. Applications are made to the evaluation of some definite integrals. *H. Pollard.*

Loomis, Lynn H. A note on the Hilbert transform. Bull. Amer. Math. Soc. **52**, 1082–1086 (1946).

A new and simple real variable method is used to deal with

$$\tilde{f}(x)=P\int_{-\infty}^{\infty}(t-x)^{-1}dF(t),$$

where P denotes the principal value and

$$V(F)=\int_{-\infty}^{\infty}|dF(t)|<\infty.$$

The author proves the Plessner-Pollard existence theorem and improves upon Kolmogoroff's result [Fund. Math. 7, 24–29 (1925)], showing that, if $V(F)$ is finite and $M>0$, then $\tilde{f}(x)$ exists almost everywhere and the measure of the set where $\tilde{f}(x)>M$ is at most $16M^{-1}V(F)$. The proof is based on a lemma concerning the intervals where $\sum_{j=1}^{n}c_j(x-a_j)^{-1}-M$ $(c_j>0)$ is positive. By rearranging $\tilde{f}(x)$, the author shows that, if $0<p<1<p+q$, then $|\tilde{f}(x)|^p(1+|x|)^{-q}\in L_1$. For the special case when $F(t)$ is absolutely continuous a more detailed result was proved by the reviewer [J. London Math. Soc. 18, 66–71 (1943); these Rev. 5, 96]. *H. Kober* (Birmingham).

Polynomials, Polynomial Approximations

Reutter, Fritz. Die Werteverteilung ganzer rationaler Funktionen. Jber. Deutsch. Math. Verein. **51**, 258–282 (1941).

The following result is here obtained as an immediate application of a theorem due to Siebeck [J. Reine Angew. Math. 64, 175–182 (1865)], and of remarks due to Haenzel [S.-B. Berlin. Math. Ges. 27, 16–19 (1928)]. Given a polynomial $f(z)$ of degree n and the constant a, the complete n-gon with vertices at the points $z_1, \cdots, z_n$, where $f(z_p)=a$, is, at the mid-points of its sides, tangent to a member of a family of confocal algebraic curves of class $n-1$, which has its $n-1$ real foci at the zeros of the derivative $f'(z)$. This result is then studied in the particular cases $n=3$ and 4 and also generalized to entire functions of Weierstrass type.

M. Marden (Milwaukee, Wis.).

de Bruijn, N. G. On the zeros of a polynomial and of its derivative. Nederl. Akad. Wetensch., Proc. **49**, 1037–1044 = Indagationes Math. **8**, 635–642 (1946).

The author proves, among others, the following theorem. Let $f(x)$ be a polynomial with real coefficients, $d_1, \cdots, d_n$ its roots, $\beta_1, \cdots, \beta_{n-1}$ the roots of $f'(x)$. Then

$$n^{-1}\sum_{j=1}^{n}|\mathfrak{J}(d_j)|\geqq(n-1)^{-1}\sum_{j=1}^{n-1}|\mathfrak{J}(\beta_j)|,$$

with equality only if all the roots are real. *P. Erdös.*

Merli, L. Sulla approssimazione delle funzioni continue mediante polinomi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 1175–1180 (1946).

Let $x_k^{(n)}=\cos\{(2k-1)\pi/2n\}$ $(n=1, 2, \cdots; k=1, 2, \cdots, n)$ and let $l_k^{(n)}(x)$ be the fundamental Lagrange polynomials corresponding to the Chebyshev abscissas $x_k^{(n)}$. It was shown by Grünwald [Acta Math. 75, 219–245 (1943); these Rev. 7, 157] that, if $f(x)$ is continuous for $-1\leqq x\leqq 1$, the polynomials $G_n(f; x)=\sum_{k=1}^{n}f(x_k^{(n)})\{l_k^{(n)}(x)\}^2$, of order $2n-2$ and coinciding with $f(x)$ at the points $x_1^{(n)}, \cdots, x_n^{(n)}$, converge to $f(x)$ in the interior of $(-1, 1)$ and that the convergence is uniform in every interval $(-1+\epsilon, 1-\epsilon)$. The author shows that, under the same assumptions concerning $f(x)$, the polynomials $G_n(h; x)$, where $h(x)=(1-x^2)^{\frac{1}{2}}f(x)$, converge uniformly to $h(x)$ in the closed interval $(-1, 1)$.

A. Zygmund (Philadelphia, Pa.).

Cinquini, Silvio. Sopra alcuni risultati relativi al problema dell'approssimazione delle funzioni. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **6(75)**, 23–36 (1942).

Let $f(x)$ and $g(x)$ be two integrable functions and let (P_n) and (Q_n) be the corresponding sequences of Stieltjes

polynomials approximating f and g, respectively. Then it is known that, if $|f|^p$ and $|g|^q$ are integrable, where $p+q=pq$, the following formulas hold:

$$\text{(1)}\qquad \lim_{n} \int |g||f-P_n|dx=0,$$

$$\text{(2)}\qquad \lim_{m,n} \int |fg-P_nQ_m|dx=0,$$

$$\text{(3)}\qquad \lim_{m,n} \int |f-P_n||g-Q_m|dx=0.$$

The author generalizes these conditions on f and g, for the case of formula (3), by replacing the special functions $H(u)=u^p/p$, $K(u)=u^q/q$, by an arbitrary pair of functions H and K with the properties: $H(0)=K(0)=H'(0)=K'(0)=0$, $H'(u)$ and $K'(u)$ are continuous and increasing for $u\geqq 0$, and $H'(K'(u))\equiv u$. For the case of formula (2), it is assumed in addition that there is a continuous nonnegative function $\Psi(u)$ such that $\lim_{u\to+\infty}\Psi(u)/u=+\infty$ and $H(\Psi(|f|))$ and $K(\Psi(|g|))$ are integrable. For the case of formula (1) it is assumed that $H(|f|)$ and $K(\Psi(|g|))$ are integrable. When $H(u)=u^p/p$, $K(u)=u^q/q$, and $H(|f|)$ and $K(|g|)$ are integrable, there always exists such a function Ψ, so that the author's theorems are true generalizations of the earlier ones. The existence of Ψ may be derived from Nagumo's criterion for uniform absolute continuity, according to which there exists a function Φ which is such that $\Phi(|f|^p)$ and $\Phi(|g|^q)$ are integrable and has other properties similar to those required for Ψ. Then we may take $\Psi(u)$ equal to the lesser of $\{\Phi(u^p)\}^{1/p}$ and $\{\Phi(u^q)\}^{1/q}$. The author concludes with some results for the derivatives of f, g, P_n and Q_m, including formulas corresponding to (1), (2) and (3).

L. M. Graves (Chicago, Ill.).

Bernstein, S. Généralisation d'un résultat de S. M. Nikolsky. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 583–585 (1946).

The author deals with the best approximation $E_n(f(x))$ of functions $f(x)$, belonging to Lip α $(0<\alpha\leqq 1)$ by polynomials of degree n on $-1\leqq x\leqq 1$. Nikolsky [same C. R. (N.S.) **52**, 7–9 (1946); these Rev. **8**, 266] proved that, if $|f(x+h)-f(x)|\leqq|h|$, $E_n(f(x))\leqq(n+1)^{-1}c_n$, where c_n is the best possible constant, $c_n\leqq\pi/2$ and $c_n\to\pi/2$ as $n\to\infty$. The author treats the case $0<\alpha\leqq 1$ and proves a theorem comparing the best approximations by polynomials of degree n, by trigonometric sums of order n and by entire functions of exponential type; Nikolsky's result follows at once. His proof is based on results of his former papers [same C. R. (N.S.) **51**, 331–334; **52**, 563–566 (1946); these Rev. **8**, 20, 323].

H. Kober (Birmingham).

Colombani, A. La théorie des filtres électriques et les polynomes de Tchebichef. J. Phys. Radium (8) **7**, 231–243 (1946).

This deals with the electrospherical polynomials [A. Guillet and M. Aubert, Propriétés électrostatiques des systèmes sphériques, Mémor. Sci. Phys., no. 38, Gauthier-Villars, Paris, 1938; M. Parodi, Applications des polynomes électrosphériques à l'étude des systèmes oscillants à un grand nombre de degrés de liberté, Memor. Sci. Phys., no. 47, Gauthier-Villars, Paris, 1944; these Rev. **7**, 295] and their application to the theory of electric wave filters composed of identical iterated networks. It differs from the work of Parodi in emphasizing the close connection between these polynomials and the classical trigonometric polynomials of Chebyshev. It is shown that all the required properties of the electrospherical polynomials are easily derived from well-known properties of the Chebyshev polynomials. These properties are derived anew. The paper also contains a table of numerical values of the first ten electrospherical polynomials for 41 equally spaced values of the argument and a similar table for a set of related polynomials.

O. Frink (State College, Pa.).

Koschmieder, Lothar. Eine Entwicklung nach Produkten Gegenbauerscher Polynome. Akad. Wiss. Wien, S.-B. IIa. **151**, 141–146 (1942).

Generalising a result by T. G. Cowling [Quart. J. Math., Oxford Ser. **11**, 222–224 (1940); these Rev. **2**, 283] the author proves that

$$(x^2+y^2-1)^{\frac{1}{2}n}C_n^\nu(xy(x^2+y^2-1)^{-\frac{1}{2}})$$
$$=\frac{\Gamma(n+2\nu)}{\Gamma(n+1)}\sum_{r=0}^{n}\frac{\Gamma(r+1)}{\Gamma(r+2\nu)}A_{nr}^\nu C_r^\nu(x)C_r^\nu(y),$$

where $C_n^\nu(x)$ is the Gegenbauer polynomial and the A_{nr}^ν are the coefficients in the expansion $x^n=\sum A_{nr}^\nu C_r^\nu(x)$.

A. Erdélyi (Pasadena, Calif.).

Bottema, O. On a generalisation of the formula of Hille and Hardy in the theory of Laguerre polynomials. Nederl. Akad. Wetensch., Proc. **49**, 1032–1036 = Indagationes Math. **8**, 630–634 (1946).

The author sums the series

$$\sum_{n=0}^{\infty}\frac{n!\Gamma(n+\alpha+k+1)}{\Gamma^2(n+\alpha+1)}t^n(xy)^{\frac{1}{2}\alpha}e^{-\frac{1}{2}(x+y)}L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)$$

in which k is a nonnegative integer and $L_n^{(\alpha)}(x)$ the generalised Laguerre polynomial. For $k=0$ the sum is known and is expressed in terms of Bessel functions by the so-called Hille-Hardy formula. The general result is obtained by writing down the Hille-Hardy formula, multiplying by $t^{\alpha+k}$, differentiating k times and then multiplying by $t^{-\alpha}$. "The proof is elementary. . . . The only difficulty arises from the arrangement of the right hand member. Once the formula is discovered, the proof can be best given by induction."

A. Erdélyi (Pasadena, Calif.).

✱Sansone, Giovanni. I polinomi di Hermite e di Laguerre come autosoluzioni. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 126–127. Edizioni Cremonense, Rome, 1942.

Announcement of results published in Boll. Un. Mat. Ital. (2) **2**, 193–200 (1940); these Rev. **2**, 43.

Thiruvenkatachar, V. R. On a polynomial arising in the theory of elementary particles. Proc. Indian Acad. Sci., Sect. A. **25**, 67–69 (1947).

The polynomials

$$\eta_s(x)=\frac{1}{(2s)!}\left\{\prod_{q=-s}^{s}(x+q)\right\}\sum_{r=0}^{2s}\binom{2s}{r}\frac{1}{x-s+r},$$
$$s=m/2;\ m=0,1,2,\cdots,$$

occur (the author states) in a problem dealing with the spin of elementary particles. The present note has for object the derivation of a second order recurrence relation for $\eta_s(x)$.

Write $y=2x$, $\eta_s(x)=\phi_m(y)$. Then $\phi_m(y)=\psi_m(y)I_m(y)$, where

$$\psi_m(y)=\frac{1}{2^m m!}(y-m)(y-m+2)\cdots(y+m-2)(y+m);$$

$$I_m(y)=\sum_{r=0}^{m}\binom{m}{r}\frac{1}{y-m+2r};$$

$\psi_m(y)$ satisfies the recursive formula

$$\text{(a)}\qquad \psi_{m+2}(y)=\frac{y^2-(m+2)^2}{4(m+1)(m+2)}\psi_m(y).$$

Also,

$$I_m(y)=\int_0^1 u^{y-m-1}(1+u^2)^m du, \qquad y-m>0,$$

from which, by two separate integrations by parts, one obtains the recursion

$$\text{(b)}\qquad I_{m+2}(y)=\frac{1}{y^2-(m+2)^2}\{2^{m+2}y-4(m+1)(m+2)I_m(y)\}.$$

Formulas (a) and (b) then lead to a recursion formula for $\phi_m(y)$, which converts to the desired one for $\eta_s(x)$:

$$(2s+3)(2s+4)\eta_{s+2}(x) +\{(2s+3)(2s+4)+(2s+2)^2-4x^2\}\eta_{s+1}(x) +\{(2s+2)^2-4x^2\}\eta_s(x)=0.$$

A list is provided of the early values of $\eta_s(x)$: $s=m/2$, $m=0, 1, \cdots, 7$. *I. M. Sheffer* (State College, Pa.).

Thijsen, W. P. On Lubbock's polynomials. Nieuw Arch. Wiskunde (2) 22, 159–161 (1946).

Polynomials $P_i(t)$ are defined by means of

$$\{(1+z)^{1/m}-1\}^{-1}=\frac{m}{z}+\frac{m-1}{2}+\sum_{i=2}^{\infty}\frac{P_i(m^2)}{i!m}z^i,$$

where

$$P_{2k-1}(m^2)=a_{2k-1,0}m^{2k}+a_{2k-1,1}m^{2k-2}+\cdots+a_{2k-1,k},$$
$$P_{2k}(m^2)=a_{2k,0}m^{2k}+a_{2k,1}m^{2k-2}+\cdots+a_{2k,k}.$$

It is proved that

$$a_{2k-1,i}=(k/i)B_{2i}C_{2k-2i}^{2k-1},\quad a_{2k,i}=-((2k+1)/2i)B_{2i}C_{2k-2i+1}^{2k},$$

where the B_{2i} are the Bernoulli numbers and the C_i^{2k} are defined by $x(x-1)\cdots(x-k+1)=\sum_{i=0}^{k-1}(-1)^iC_i^kx^{k-i}$.
L. Carlitz (Durham, N. C.).

Special Functions

Pidduck, F. B. Lommel's functions of small argument. Quart. J. Math., Oxford Ser. 17, 193–196 (1946).

The function

$$I_{mn}(z)=(m+n-1)zJ_n(z)S_{m-1,n-1}(z)-zJ_{n-1}(z)S_{m,n}(z)$$

is the indefinite integral of $z^mJ_n(z)$ [Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, 1922, § 10.74(3)]. If the lower limit of the corresponding definite integral is zero, $I_{mn}(0)$ is wanted and this requires special investigation. From the known expansions of Bessel functions and Lommel functions $I_{mn}(0)$ can be obtained when $m\neq 1$ and $n=0$, and when $m-n$ is not an odd negative integer and $n>1$. The other cases are considered under four heads: (a) $m=1$, $n=0$; (b) $m=0$, $n=1$; (c) $m=n-1$ generally; (d) $m=n-2p-1$, where p is a positive integer. The results are applied to obtain approximate values for the integral

$$E_n^s(x)=\int_0^\infty J_0(xy)e^{iny}y^sdy$$

which arises in the theory of currents in an aerial parallel to the earth. *A. Erdélyi* (Pasadena, Calif.).

Meijer, C. S. On the G-function. V, VI. Nederl. Akad. Wetensch., Proc. 49, 765–772, 936–943 = Indagationes Math. 8, 468–475, 595–602 (1946).

[For the first four parts see the same Proc. 49, 227–237, 344–356, 457–469, 632–641 = Indagationes Math. 8, 124–134, 213–225, 312–324, 391–400 (1946); these Rev. 8, 156.] After more preliminary lemmas follows the fourth expansion formula ("the most important theorem of the present paper") which expresses $G_{p,q}^{m,n}$ in terms of ν functions $G_{p,q}^{k,l}$, μ functions $G_{p,q}^{k,l-1}$ and $k-\mu-\nu$ other functions of the same type; $l\geqq 1$, $q\geqq 1$, $0\leqq m\leqq k\leqq q$, $0\leqq n-l+1\leqq\nu\leqq k$, $l+\nu-1\leqq p\leqq q$, $0\leqq\mu\leqq k-\nu$. A special case of this is a transformation formula which expresses $G_{p,q}^{m,n}$ in terms of k functions $G_{p,q}^{k,n}(0\leqq m\leqq k\leqq q)$. This is applied to Bessel and Whittaker functions and yields formulae like

$$K_\nu(z)=\frac{\sin(\lambda+1)\nu\pi}{\sin\nu\pi}K_\nu(ze^{-\lambda\pi i})-\frac{\sin\lambda\nu\pi}{\sin\nu\pi}K_\nu(ze^{-(\lambda+1)\pi i}),$$

where λ is any integer. The particular cases $l=1$, $k=q$ of the four principal expansions are established in part V and used in part VI to attack the main problem of the paper. It is shown that by means of these expansions $G_{p,q}^{m,n}(z)$ $(q>p)$ can be expressed as a linear combination of the fundamental solutions valid at infinity of the appropriate differential equation, and this again enables the author to investigate for all values of m, n, p, q $(p<q)$ and of arg z the asymptotic behaviour of $G_{p,q}^{m,n}(z)$ as $|z|\to\infty$. *A. Erdélyi.*

Meijer, C. S. On the G-function. VII, VIII. Nederl. Akad. Wetensch., Proc. 49, 1063–1072, 1165–1175 = Indagationes Math. 8, 661–670, 713–723 (1946).

[See the preceding review.] The investigation of the asymptotic behaviour of $G_{p,q}^{m,n}(z)$ for large values of $|z|$ and arbitrary values of arg z is continued in part VII and completed in part VIII. The author next turns to the discussion of the analytic continuation of $G_{p,p}^{m,n}(z)$ outside the unit circle. This analytic continuation is achieved by a representation of $G_{p,p}^{m,n}(z)$ as a linear combination of the p functions

$$z^{-1+a_t}{}_pF_{p-1}\left[\begin{matrix}1+b_1-a_t, & \cdots, & 1+b_p-a_t;\\ 1+a_1-a_t, & \cdots*\cdots, & 1+a_p-a_t;\end{matrix}\; e^{(m+n-p)\pi i}z^{-1}\right],$$

$t=1, \cdots, p$, in which the asterisk indicates omission of $1+a_t-a_t$. In the last two sections the general results are applied in order to obtain the asymptotic representation of the generalised hypergeometric function ${}_pF_q(z)$ and of Whittaker's function $W_{k,m}(z)$ for large values of $|z|$ and arbitrary values of arg z. *A. Erdélyi.*

Parodi, Maurice. Sur deux applications de la correspondance symbolique

$$\mathfrak{L}[f(x^2)]=\frac{p}{\sqrt{\pi}}\int_0^\infty e^{-\frac{p^2\lambda^2}{4}}\varphi\left(\frac{1}{\lambda^2}\right)d\lambda.$$

Revue Sci. 84, 162–163 (1946).

The formula shown in the title is applied to each of the functions x^k and $\log x$ to obtain short derivations of two known properties of the gamma function. The transformation denoted by $\mathfrak{L}$ here is the product of the parameter p

by the Laplace transformation; $\phi(p)$ is the corresponding transform. *R. V. Churchill* (Ann Arbor, Mich.).

Harmonic Functions, Potential Theory

Sire, Jules. Les fonctions harmoniques absolument continues. Ann. Univ. Lyon. Sect. A. (3) 6, 5–18 (1943).

Let T be a domain whose boundary S is bounded and also reduced, that is, such that each spherical portion is of nonzero capacity. Let F be a closed set of S and denote by r_P the distance of F from an arbitrary point P of space. Set $\alpha_n(P)=(1+nr_P)^{-1}$, $n=1, 2, \cdots$. The sequence $\{\alpha_n(P)\}$ converges uniformly in the interior of T to a nonnegative harmonic function $\alpha(P)$ which is the greatest lower bound of all functions harmonic in T with boundary values 1 on F and nonnegative on $S-F$. The author defines the weight of F as the function $\alpha(P)$. The definition is extended to apply to open sets. Exterior and interior weight is defined for general sets; sets for which these two weights are equal are called normal and have properties corresponding to those of Lebesgue measurable sets. For point functions normal on S integrals analogous to ordinary Lebesgue integrals are studied.

With a given $F(P)$ harmonic in T the author associates a certain nondecreasing sequence $\{\Phi_m(P)\}$ of harmonic functions. If $\{\Phi_m(P)\}$ diverges $F(P)$ is said to be of unbounded variation. If $\{\Phi_m(P)\}$ converges the limit $\Phi(P)$ is a nonnegative harmonic function in T and is by definition the total variation of $F(P)$. Positive and negative variations of $F(P)$ are also defined. A theory is developed analogous to that for functions of a single variable; nonnegative functions play the rôle of monotone increasing functions of a single variable. *F. W. Perkins* (Hanover, N. H.).

Sansone, G. Condizioni sufficienti di esistenza e limitazioni delle derivate normali al contorno nel problema di Dirichlet per un cerchio. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 1042–1045 (1946).

Let $f(\theta)$ be the boundary function of a harmonic function $u(\rho,\theta)$, $0\leqq\rho<1$, $0\leqq\theta\leqq 2\pi$; let $k\geqq 1$; if $f^{(k)}(\theta)$ satisfies a Lipschitz condition on the unit circle then $\partial^k u(\rho,\theta)/\partial\rho^k$ exists as $\rho\rightarrow 1$, for all θ. An estimate is also given in terms of the boundary function. The proof is elementary.

O. Szász (Cincinnati, Ohio).

Landkof, N. On some characteristics of irregular points in the generalized problem of Dirichlet. Rec. Math. [Mat. Sbornik] N.S. 19(61), 175–182 (1946). (Russian. English summary)

The author considers some functions which may serve for characterization of an irregular point P of an open set Ω. Let $c(\rho)$ be the capacity of the intersection $\omega(\rho)$ of the sphere $s(\rho)$ of radius ρ about P with the exterior of Ω; let $w(\rho)$ be the value at P of the potential of equilibrium mass distribution over $\omega(\rho)$; let $\mu(\rho)$ be the part of Green's masses for the point P lying within $s(\rho)$. It is shown that

$$\varliminf_{\rho\to 0}\frac{\log\mu(\rho)}{\log\rho}=\varliminf_{\rho\to 0}\frac{\log w(\rho)}{\log\rho}=\varliminf_{\rho\to 0}\frac{\log c(\rho)}{\log\rho}-1.$$

The common value of the above limits is considered as the order of the given irregular point. *E. F. Beckenbach.*

Evans, G. C. A necessary and sufficient condition of Wiener. Amer. Math. Monthly 54, 151–155 (1947).

A lecture on capacity in potential theory.

Vekua, Ilia. Modification of an integral transformation and some of its properties. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 6, 177–183 (1945). (Georgian and Russian)

The author studies the equation (1) $\Delta u+\lambda^2 u=0$ (λ real) in a domain T in Euclidean n-space, star-shaped with respect to the origin. The author previously established [same Bull. 4, 385–392, 843–852, 941–950 (1943); these Rev. 6, 154, 177] that all regular solutions of (1) (i.e., those with continuous first and second derivatives) in T are given by (2) $u=M_\lambda u_0$, where M_λ is a Volterra operator (involving Bessel's function J_0) and u_0 is any function harmonic in T. The author establishes explicitly the inversion formula of (2): $u_0=M_\lambda^{-1}u$ (the inversion is unique). An application is made to a boundary value problem related to an infinite star T. *W. J. Trjitzinsky* (Urbana, Ill.).

Deny, Jacques. Sur les infinis d'un potentiel. C. R. Acad. Sci. Paris 224, 524–525 (1947).

Let μ be a distribution of positive mass in Euclidean n-space; then $U(M)=\int(MP)^{2-n}d\mu(P)$ is the corresponding potential function. If $n=2$, then $U(M)=-\int\log(MP)d\mu(P)$. In a recent paper [Bull. Soc. Math. France 73, 74–106 (1945); these Rev. 7, 447], H. Cartan has shown that the set $E=[M\mid U(M)=+\infty]$ is a G_δ and, if $U(M)\not\equiv+\infty$, then E is of zero exterior capacity. In this note the author proves that these properties characterize the set E; he proves that, if E is a G_δ of zero exterior capacity, then there is a distribution of positive mass μ such that the potential is infinite at each point of E and finite at each point not of E. This result complements earlier results due to Evans [Monatsh. Math. Phys. 43, 419–424 (1936)] and Privaloff [Rec. Math. [Mat. Sbornik] N.S. 12(54), 85–90 (1943); these Rev. 4, 278]. *M. O. Reade* (Ann Arbor, Mich.).

Caldonazzo, Bruto. Considerazioni geometriche sui potenziali gravitazionali. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 6(75), 239–259 (1942).

The author considers the gravitational potential u of a mass distributed in a field S with density ρ, ρ being a differentiable function of the coordinates. The field S is bounded by a regular surface Σ which belongs to the system of surfaces $u=$constant. The orthogonal trajectories are the lines of force. As is well known, the second derivatives of u are discontinuous in passing the surface Σ. Inside Σ they satisfy Poisson's equation $\Delta u=-4\pi\rho$; outside, Laplace's equation $\Delta u=0$. Denoting the modulus of grad u by g and putting $\sigma=du/dP$, the author investigates the discontinuities of grad g, of the third derivatives of u, of the derivatives of σ, and those of the lines of force, and, furthermore, the influence of those discontinuities on the differential elements of the equipotential surfaces. He proves that the total curvature and its derivatives are continuous.

H. Bremekamp (Delft).

***Somigliana, C. Il campo gravitazionale della terra.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 72–87. Edizioni Cremonense, Rome, 1942.

Expository lecture.

✶Boaga, G. Le anomalie gravimetriche e le deviazioni della verticale per pianeti sferoidici non di rotazione. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 534–540. Edizioni Cremonense, Rome, 1942.

Amerio, Luigi. Sull'integrazione dell'equazione $\Delta_4 u=0$ in due variabili. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **8(77)**, 377–419 (1944).

The equation $\Delta\Delta u=0$, Δ being the Laplace operator in two variables, is transformed into $\partial^4 u/\partial z^2\partial\bar{z}^2=0$ by the substitution $z=x+iy$, $\bar{z}=x-iy$ and so the general solution is (1) $u(z,\bar{z})=F(z)+G(\bar{z})+\bar{z}H(z)+zK(\bar{z})$, where F, G, H, K are arbitrary analytic functions of the variables z or $\bar{z}$. The determination of a solution valid in a simply connected domain bounded by a closed curve S and satisfying given boundary conditions on S requires the determination of the functions F, G, H, K. If the function $z=\varphi(\tau)$ giving the conformal representation of the interior of S on the circle $|\tau|<1$ is known, the solution is immediate if u and Δu or $\partial\Delta u/\partial n$ are given on S. The classical problem, where u and $\partial u/\partial n$ are given on S, is more difficult. Hadamard [Mém. Acad. Sci. Inst. France (2) **33**, no. 4 (1908), p. 9] reduced it to the solution of a system of integral equations without using the function φ. N. Muschelišvili [Math. Ann. **107**, 282–312 (1932)], using the function φ, gave another system. In the present paper a single equation is deduced equivalent to the latter system. The data on S are equivalent to giving $(\partial u/\partial\bar{z})_S=A(\tau)$ and $(\partial u/\partial z)_S=B(\tau)$. If (1) is the required solution the functions F, G, H, K are to be derived from the differential equations

$$\text{(2)}\qquad \begin{aligned} &h(\tau)+\varphi(\tau)k'(\tau)/\psi'(\tau)+g'(\tau)/\psi'(\tau)=A(\tau),\\ &k(\tau)+\psi(\tau)h'(\tau)/\varphi'(\tau)+f'(\tau)/\varphi'(\tau)=B(\tau),\end{aligned}$$

where $\psi(\tau)=\varphi(\tau)$, $f(\tau)=F(\varphi(\tau))$ and $h(\tau)=H(\varphi(\tau))$ are continuous and have continuous first derivatives for $|\tau|\leqq 1$, $g(\tau)=G(\psi(\tau))$ and $k(\tau)=K(\psi(\tau))$ for $|\tau|\geqq 1$. Putting $\omega(t)=v'(t)/\varphi'(t)+w'(t)/\psi'(t)$, where v is linearly dependent on φ and h, w on ψ and k, the author shows that the system (2) is equivalent to the integral equation

$$\omega(t)+\frac{\lambda}{2\pi i}\int_{|\tau|=1}\left\{\frac{1}{\varphi'(t)}\frac{\partial}{\partial t}\frac{\varphi(\tau)-\varphi(t)}{\tau-t}+\frac{1}{\tau^2\psi'(t)}\frac{\partial}{\partial t}\frac{\psi(\tau)-\psi(t)}{\tau^{-1}-t^{-1}}\right\}\omega(\tau)\partial\tau=\frac{A_1'(t)}{\varphi'(t)}+\frac{B_2'(t)}{\psi'(t)}$$

for $\lambda=1$; he proves that $\lambda=1$ is not a characteristic value of the equation. The solution is to be found by successive approximations. Finally, for the case that $\varphi(\tau)$ is a rational function a simpler procedure is given for the construction of the solution, profiting by a general remark on the position and the nature of the singularities of the functions v and w. *H. Bremekamp* (Delft).

Dolidze, D. On the limit values of the hydrodynamical Green's function. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **6**, 753–761 (1945). (Georgian and Russian)

In previous papers the author [same Bull. **4**, 11–16 (1943); **5**, 373–382 (1944); these Rev. **8**, 104] considered the "hydrodynamical Green's function" $G(P,Q,t)$ of a simply connected plane domain D which satisfies the equation $\nu\Delta\Delta G-\partial\Delta G/\partial t=0$ for $P\neq Q$, $t>0$, possesses prescribed singularities for $P=Q$ and for $t=0$ and satisfies the boundary condition (∗) $G=\partial G/\partial n=0$ $(t>0)$ on the (sufficiently regular) curve C bounding D. (Here ν is a constant, $\partial/\partial n$ denotes differentiation in the direction normal to C.) In this note he shows that

$$\lim_{t\to\infty}\int_0^t G(P,Q,\tau)d\tau$$

exists and is equal to Green's function of the biharmonic equation $\Delta\Delta u=0$ for the domain D and the boundary condition (∗). *L. Bers* (Syracuse, N. Y.).

Differential Equations

Wintner, Aurel. On the Laplace-Fourier transcendents occurring in mathematical physics. Amer. J. Math. **69**, 87–98 (1947).

A classical method of Laplace for solving an ordinary linear differential equation consists in assuming a solution of the form $\int_C e^{xt}v(t)dt$, where $v(t)$ satisfies the adjoint equation and C is a suitably chosen contour. For the applications it is preferable if C is a line segment and $v(t)$ is positive. The author sets himself the problem of deciding a priori when such a restricted solution exists in the case of the normalized second order equation $w''+f(x)w=0$. He proves the following results. (I) If $f(1/z)$ is an entire function in z and if $f(iy)$ is completely monotone for $0<y$, then the equation has a Fourier solution $w(x)=\int_0^\infty e^{ixt}d\phi(t)$, where $\phi(t)$ is monotone but not necessarily bounded and the integral exists as an Abel limit for $x>0$. (II) If $f(x)$ is negative and completely monotone on the positive real axis, i.e., $(-1)^n f^{(n)}(x)\leqq 0$ for all n, then the equation has a completely monotone solution $w(x)=\int_0^\infty e^{-xt}d\phi(t)$. Various intermediary results are established. Thus if $f(x)$ is of class $C^{(m)}$ and if $(-1)^n f^{(n)}(x)\leqq 0$, $n=0, 1, \cdots, m$, then the equation has a solution $w(x)$ $(\neq 0)$ with $(-1)^n w^{(n)}(x)\geqq 0$, $n=0, 1, \cdots, m+1$. In connection with (II) the author points out, refuting an assertion of A. Kneser, that $\lim_{x\to\infty} w(x)$ may be positive, but only if $f(x)\in L(0,\infty)$. The same condition is also necessary in order that the equation $w''+f(x)w=0$, $f(x)>0$ for $x>0$, shall have nonoscillatory solutions. [Reviewer's remark. The condition is not sufficient in either case. A necessary and sufficient condition that $w''\pm f(x)w=0$, $f(x)>0$, have a solution with $\lim_{x\to\infty} w(x)=1$ is that $xf(x)\in L(0,\infty)$. A. Wiman [Ark. Mat. Astr. Fys. **12**, no. 14 (1917), pp. 4–5] showed that, if $\lim_{x\to\infty} x^2 f(x)=\gamma$, then the solutions of $w''+f(x)w=0$ are oscillatory for $\gamma>\frac{1}{4}$ and nonoscillatory for $\gamma<\frac{1}{4}$. This result can be extended to a logarithmic scale of nonoscillation criteria.] *E. Hille.*

✶Sansone, Giovanni. Studi asintotici sulle equazioni differenziali lineari nel campo reale. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 39–55. Edizioni Cremonense, Rome, 1942.

Expository lecture.

✶Graffi, Dario. Sul calcolo degli autovalori per una corda non omogenea. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 353–359. Edizioni Cremonense, Rome, 1942.

✶Laura, E. Sulle piccole oscillazioni di una superficie flessibile inestendibile intorno ad una posizione di equilibrio stabile. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 346–352. Edizioni Cremonense, Rome, 1942.

Grünwald, Erich. Lösungsverfahren der Laplace-Transformation für Ausgleichsvorgänge in linearen Netzen, angewandt auf selbsttätige Regelungen. Arch. Elektrotechnik **35**, 379–400 (1941).

This paper discusses transient phenomena in automatic regulators, using Laplace transform methods to handle the associated systems of ordinary linear differential equations. After a general discussion of these methods, the known results for any network (electrical, mechanical, thermal or otherwise) are systematized and generalized in dimensionless form and stability criteria derived. Application is then made to the analysis of a speed regulator for a steam turbine, showing how the solution of equations of higher degree may be avoided by harmonic analysis. The voltage regulation of an a.c. generator is next discussed, giving approximate methods of solution. The paper ends with some discussion of the inverse problem of designing a regulator having prescribed transient behavior.

L. C. Hutchinson (Brooklyn, N. Y.).

Weiss, Herbert K. Analysis of relay servomechanisms. J. Aeronaut. Sci. **13**, 364–376 (1946).

The literature on servomechanisms has concentrated on the continuous type as more amenable to mathematical analysis. The present paper takes a step in the direction of analyzing relay servomechanisms where the control is discontinuous. The author applies to simple relay servomechanisms a method previously employed for other purposes [K. J. DeJuhasz, J. Appl. Mech. **12**, 175–177 (1945); N. Minorsky, J. Franklin Inst. **240**, 25–46 (1945); these Rev. **7**, 14], namely, the representation of the motion in the position-velocity or "phase" plane, where the discontinuities involved divide the plane into regions (generally bounded by straight lines) within each of which the motion can be determined by elementary means. The possibilities of the method are indicated. Practical and graphical methods are devised for determining the transient and steady state solutions. Exact solutions are obtained for particular cases of interest and the problem of hunting is discussed with the techniques developed. *L. C. Hutchinson.*

Horvay, Gabriel. Unstable solutions of a class of Hill differential equations. Quart. Appl. Math. **4**, 385–396 (1947).

The author starts from the differential equation

$$\frac{d^2v}{d\psi^2}+\{\theta_{-2}e^{-2i\psi}+\theta_{-1}e^{-i\psi}+\theta_0+\theta_1e^{i\psi}+\theta_2e^{2i\psi}\}v=0,$$
$$\theta_0^*=\theta_0,\quad \theta_{-1}^*=\theta_1,\quad \theta_{-2}^*=\theta_2,$$
$$\theta_1=O(\mu),\quad \theta_2=O(\mu^2),\qquad \mu>0,$$

which is an extension of Mathieu's equation. This equation is solved by the well-known method of a complex Fourier series expansion with undetermined coefficients. Insertion of this series into the equation leads to a special type of Hill's determinant, from which a relation between the characteristic exponent of the solution and the parameters of the equation is obtained. The elaboration of this relationship depends on the expansion of Hill's infinite determinant. This is achieved by means of suitable sub-determinants using Laplace's expansion. This expansion is carried out numerically using the first terms of a power series. It is shown that a well-known expansion of Hill's determinant in the case of Mathieu's equation is a special case of the author's formulae. A table is given for the coefficients of the first terms of the power series and two numerical examples of the application of the series and of the table are included.

M. J. O. Strutt (Eindhoven).

Meksyn, D. Stability of viscous flow between rotating cylinders. III. Integration of a sixth order linear equation. Proc. Roy. Soc. London. Ser. A. **187**, 492–504 (1946).

[For parts I and II see the same vol., 115–128, 480–491 (1946); these Rev. **8**, 415.] The paper deals with the solutions of a differential equation

$$y^{(6)}-3\lambda^2y^{(4)}+3\lambda^4y''-\lambda^6\{1+hx\}y=0,$$

in which λ is a large parameter which, like the variable x and the constant h, is real. Forms that are the leading terms of asymptotic solutions for an equation such as this, and which are pertinent either when $1+hx$ is large and positive or numerically large and negative, are obtainable by familiar means. The so-called "connection formulas" which relate the forms of the one of these sets with those of the other are here the primary issue. The differential equation is solvable by contour integrals. From an analysis of these latter, the connection formulas are deduced. The paper purports to infer beyond this comparable results for a considerably more complicated differential equation which pertains to the stability of the flow of a viscous fluid between rotating cylinders and which is in a certain sense approximated by the equation above. It is, however, difficult for the reviewer to read from it much more than has been reported. *R. E. Langer* (Madison, Wis.).

Azevedo do Amaral, Ignacio M. On the integration of ordinary linear differential equations and integral equations. Rend. Accad. Sci. Fis. Mat. Napoli (4) **12**, 345–352 (1942). (Portuguese. Italian summary)

The results of this paper are contained in subsequent papers of the author already reviewed [for example, Anais Acad. Brasil. Ci. **16**, 261–271 (1944); see these Rev. **6**, 271, and references given there]. As pointed out previously, the author's methods are so special that the derived integral formula for a solution is not needed; in particular, if the solution of a differential equation can be reduced to an integral equation of the first kind by the proposed method, then this integral equation has a constant function as solution and the solution of the differential equation is immediate.

W. T. Reid (Evanston, Ill.).

Fichera, Gaetano. Sull'integrazione in grande delle forme differenziali esterne di qualsivoglia grado. Ricerca Sci. **16**, 1117–1119 (1946).

A necessary condition that the equation rot $\mathbf{u}=\mathbf{v}$ admits a uniform solution in a domain D is that div $\mathbf{v}=0$. This is, however, not always a sufficient condition. A necessary and sufficient condition is that there exists in D a closed surface for which $\iint(\mathbf{v}\times\mathbf{n})d\sigma$ is zero ($\mathbf{n}$ is the normal vector). This theorem is generalized for a domain D in r-space and an alternating form π of degree $k+1$, $k<r$. The necessary and sufficient condition for the integrability of π is related to topological properties discussed in the thesis of G. de Rham [J. Math. Pures Appl. (9) **10**, 115–200 (1931)].

D. J. Struik (Cambridge, Mass.).

Fichera, G. Sull'integrabilità in grande delle forme differenziali esterne di qualsivoglia grado. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 540–543 (1946).

Given the differential form of degree k,

$$\Omega=\sum_{i_1<\cdots<i_k}X_{i_1\cdots i_k}[\delta x_{i_1}\cdots\delta x_{i_k}],$$

where the X are functions of $x_1, \cdots, x_r$ and

$$[\delta x_{i_1} \cdots \delta x_{i_k}] = \left\| \begin{matrix} \delta_1 x_1 & \cdots & \delta_1 x_r \\ \cdots & & \cdots \\ \delta_k x_1 & \cdots & \delta_k x_r \end{matrix} \right\|, \qquad k \leqq r,$$

it is stated that a necessary and sufficient condition that Ω is integrable in a domain in which the X are uniform is that for a closed V_{k+1} in D

$$\int \sum_{q_1 < \cdots < q_{k+1}} A_{q_1 \cdots q_{k+1}} [\delta x_{q_1} \cdots \delta x_{q_{k+1}}] = 0,$$

where

$$A_{q_1 \cdots q_{k+1}} = \sum_{h=1}^{k+1} (-1)^{h-1} \frac{\partial}{\partial x_{q_h}} X_{q_1 \cdots q_{h-1} q_{h+1} \cdots q_{k+1}}.$$

D. J. Struik (Cambridge, Mass.).

Tonolo, Angelo. Sulla riducibilità delle forme differenziali. Atti Mem. Accad. Sci. Padova. Mem. Cl. Sci. Fis.-Mat. (N.S.) **59**, 57–64 (1943).

L'auteur se propose d'expliciter des conditions nécessaires et suffisantes pour qu'il existe une transformation d'une forme différentielle donnée, de degré m, de n variables, en une forme analogue de $n-1$ variables. *B. Levi.*

Pfeiffer, G. Sur les équations, systèmes d'équations semi-mixtes aux dérivées partielles du premier ordre à plusieurs fonctions inconnues. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 299–301 (1946).

The author sketches two rules for integration of systems of "semi-mixed" partial differential equations of the first order with several unknown functions. *D. J. Struik.*

Pfeiffer, G. Sur les équations, systèmes d'équations semi-jacobiens, semi-jacobiens généralisés aux dérivées partielles de premier ordre à plusieurs fonctions inconnues. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 659–661 (1946).

The author sketches two ways of applying his method of integration [Bull. Acad. Sci. Ukraine 1, 41–51 (1923); C. R. Acad. Sci. Paris **176**, 62–64 (1923)] to systems of generalized semi-Jacobian equations of the first order with several unknown functions. *D. J. Struik* (Cambridge, Mass.).

Mychkis, A. Sur les domaines d'unicité pour les solutions des systèmes d'équations linéaires aux dérivées partielles. Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 489–522 (1946). (Russian. French summary)

Let there be given a system of linear partial differential equations

$$(1) \qquad \frac{\partial u_\nu}{\partial x} = \sum_{\mu=1}^{n} f_{\nu\mu}(x, y) \frac{\partial u_\mu}{\partial y} + \sum_{\mu=1}^{n} g_{\nu\mu}(x, y) u_\mu, \qquad \nu = 1, \cdots, n,$$

the coefficients $f_{\nu\mu}$, $g_{\nu\mu}$ being defined in a domain G. At every point $(x, y) \epsilon G$ a multivalued function (2) $y' = \lambda(x, y)$ is defined by the characteristic equation

$$\det \| f_{\nu\mu} - \lambda \delta_{\nu\mu} \| = 0.$$

It is assumed that λ takes on only real values. The author considers (2) as a differential equation "in contingents." A curve $y = \varphi(x)$ is a characteristic of (1) (solution of (2)) if φ is continuous and its four derivates satisfy

$$\min \lambda[x, \varphi(x)] \leqq D\varphi \leqq \max \lambda[x, \varphi(x)].$$

The theory of such differential equations [Zaremba, C. R. Acad. Sci. Paris **199**, 545–548 (1934); Bull. Sci. Math. (2) **60**, 139–160 (1936); Marchaud, C. R. Acad. Sci. Paris **199**, 1278–1280 (1934); Compositio Math. **3**, 89–127 (1936)] is the main tool of the present paper.

Holmgren [see note 1 in Hadamard, Leçons sur la Propagation des Ondes et les Équations de l'Hydrodynamique, Paris, 1903] proved that, if $\{u_\nu\}$ is a solution of (1) and $u_\nu = 0$ ($\nu = 1, \cdots, n$) on a curve l: $x = x(t)$, $y = y(t)$ violating (2) at all points, all u_ν vanish identically in a neighborhood of l. His theorem assumes analyticity of all functions considered, a restriction which was removed by Carleman [Ark. Mat. Astr. Fys. **26B**, no. 17 (1939)]. The author wants to obtain uniqueness theorems in the large from the results of Holmgren and Carleman. He considers the case when l is a vertical segment and calls a domain $G_1 \subset G$ a T-domain (with respect to l) if every characteristic extending from a boundary point of G_1 to another boundary point meets l; G_1 is called an E-domain if the uniqueness of the solution of the Cauchy problem for (1) in G_1 (with Cauchy data on l) can be inferred from the local uniqueness theorems of Holmgren and Carleman without further reference to the differential equations. The main theorem states that every T-domain is an E-domain. The converse is true under certain additional conditions (too complicated to be stated here). Examples show that these conditions are essential.

L. Bers (Syracuse, N. Y.).

Amerio, L. Sull'integrazione delle equazioni lineari di tipo ellittico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 175–182 (1946).

Let $E = \sum a_{ik} \partial^2 / \partial x_i \partial x_k + \sum b_i \partial / \partial x_i + c$ be an elliptic operator in m variables x_i, the coefficients a_{ik}, b_i, c being functions of the variables x_i; let E^* be the adjoint operator. Then if u is a solution of $E(u) = f$, Green's transformation gives

$$(*) \qquad -\int_\tau u E^*(w) d\tau = \int_\sigma \left\{ u \left(\frac{\partial w}{\partial \nu} - Lw \right) - \frac{\partial u}{\partial \nu} w \right\} d\sigma - \int_\tau f w d\tau,$$

where ν and n are the unit inward conormal and normal vectors to the surface σ bounding the volume τ, and $L = \sum \cos(nx_i) \{ b_i - \sum \partial a_{ik} / \partial x_k \}$. The problem of Dirichlet is to find the solution u which takes given values on σ.

Picone [Appunti di Analisi Superiore, A. Rondinella, Naples, 1940, pp. 752–765; these Rev. **3**, 144] gave a new method of solving the problem of Dirichlet, based on the formula $(*)$, as follows. (a) If a sequence $v_1, v_2, v_3, \cdots$ of solutions of $E^*(v) = 0$ is known and if this sequence is closed on σ, then $(*)$ determines the Fourier coefficients of $\partial u / \partial \nu$ when u is known on σ, and hence determines the boundary values of $\partial u / \partial \nu$. (b) If a sequence $w_1, w_2, w_3, \cdots$ of functions is known and if $E^*(w_r)$ is closed in τ, then, when the boundary values of u and $\partial u / \partial \nu$ are known, $(*)$ determines the Fourier coefficients of u in τ and hence determines u. The process of solution is in two steps, but Amerio points out that it can be done in one step, if we regard the unknowns, viz. u in τ and $\partial u / \partial \nu$ on σ, as components of a vector $\mathbf{G}$ and if we also regard $-E^*(w_r)$ in τ and w_r on σ as components of a vector $\boldsymbol{\omega}_r$. The inner product of the vectors $\mathbf{G}$ and $\boldsymbol{\omega}_r$ is then determined by

$$(\mathbf{G}, \boldsymbol{\omega}_r) = \int_\sigma u(\partial w_r / \partial \nu - L w_r) d\sigma - \int_\tau f w_r d\tau,$$

where the right-hand side is known.

The paper is concerned mainly with the construction of the sequences $\{v_r\}$, $\{w_r\}$, $\{\boldsymbol{\omega}_r\}$. Only a brief account is given; further details are promised in a paper to appear later. *E. T. Copson* (Dundee).

Amerio, L. Sul calcolo delle autosoluzioni dei problemi al contorno per le equazioni differenziali lineari a derivate parziali. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 352–359 (1946).

In the paper reviewed above, the solution of the problem of Dirichlet for an elliptic partial differential equation of the second order was reduced to "the problem of Picone," viz. the determination of an unknown vector **G** by the sequence of equations $(\mathbf{G}, \mathbf{z}_r)=\gamma_r$, $r=0, 1, 2, \cdots$, where $\mathbf{z}_r$ is a known vector, γ_r the known Fourier coefficients. In particular, if $(\mathbf{G}, \mathbf{z}_r)=0$ for all r, **G** is an "autosoluzione" or eigenfunction of the problem. The present paper is concerned with the determination of all the eigenfunctions of the problem of Picone for which $(\mathbf{G}, \mathbf{G})$ is finite and the application of the results to the problem of Dirichlet.

E. T. Copson (Dundee).

Amerio, L. Sul calcolo delle autosoluzioni dei problemi al contorno per le equazioni differenziali lineari a derivate parziali. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 505–509 (1946).

The results of the paper reviewed above are applied to the equation $\Delta_2 u+\lambda u=0$. *E. T. Copson* (Dundee).

✱Tolotti, Carlo. Applicazione di un nuovo metodo di M. Picone all'integrazione delle equazioni dell'elasticità in un parallelepipedo rettangolo. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 422–424. Edizioni Cremonense, Rome, 1942.

Parodi, H. Contribution à l'étude mathématique du problème du mur. J. Phys. Radium (8) 7, 287–292 (1946).

A formal method of constructing a solution $u(x, t)$ of the heat equation $u_{xx}-u_t=0$, satisfying the boundary and initial conditions $u(-a, t)=f(t)$, $u(a, t)=g(t)$, $t>0$; $u(x, 0)=F(x)$, $-a<x<a$, is given in the form $u(x, t)=E(x, t)+H(x, t)$, where $H(x, t)$ is an infinite series satisfying the initial condition and such that $\lim_{t\to\infty} H(x, t)=0$, while $E(x, t)$ satisfies the boundary conditions and is the "asymptotic solution" of the problem. Similar results are obtained for other standard boundary value problems for the heat equation.

F. G. Dressel (Durham, N. C.).

Vekua, Il'ya. On the theory of Legendre functions. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 7, 3–10 (1946). (Russian)

The paper is connected with two previous notes of the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 311–314 (1945); Appl. Math. Mech. [Akad. Nauk. SSSR. Prikl. Math. Mech.] 9, 368–388 (1945); these Rev. 8, 67]. The author considers the equation $u_{xy}+n(n+1)(1+xy)^{-2}u=0$ and shows that all its solutions which are regular in a neighborhood of the origin are:

$$u(x, y)=aP_n(z)+\int_0^1 \{xf(xt)+yg(yt)\}P_n(z+t-zt)dt,$$

where $z=(1-xy)/(1+xy)$, P_n is the Legendre polynomial, a is an arbitrary constant and f and g are arbitrary analytic functions of the respective arguments, which are regular around the origin. He later shows that the above expression is still a solution of the equation when f and g have singularities at the origin but are integrable in the sense of J. Hadamard [Le Problème de Cauchy, Hermann, Paris, 1932, p. 184]. *I. Opatowski* (Ann Arbor, Mich.).

Vekua, Il'ya. On the theory of cylinder functions. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 7, 95–101 (1946). (Russian)

The author shows that all solutions of the equation $u_{xy}+u=0$ which are regular around the origin are

$$u(x, y)=aL(xy)+\int_0^x f(t)L(xy-yt)dt+\int_0^y g(t)L(xy-xt)dt,$$

where a, f, g have the same meaning as in the paper reviewed above and $L(z)=\sum_{k=0}^{\infty}(-z)^k/(k!)^2$. He also gives other similar expressions containing successive derivatives of L and g and successive integrals of f. *I. Opatowski.*

✱Fremberg, Nils Erik. A study of generalized hyperbolic potentials with some physical applications. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] 7, 1–100 (1946).

It is well known that in the theory of Cauchy's problem for hyperbolic partial differential equations special divergence difficulties arise because the elementary solution is infinite on the "light-cone." Hadamard overcame these difficulties by introducing a new technique for determining the finite part of an infinite integral. About ten years ago, M. Riesz gave a new method of overcoming this divergence difficulty with the aid of a generalization of the Riemann-Liouville integral of fractional order and analytical continuation. The new ideas introduced by Riesz are important in themselves and also for the possibilities they offer for eliminating similar divergence difficulties in quantum electrodynamics. The present monograph gives an account of the definition of the Riesz potentials, their analytical continuation and their application to (i) the solution of the problem of Cauchy for the wave equation, (ii) the solution of the nonhomogeneous wave equation, (iii) the solution of Maxwell's equations, (iv) the solution of the equations of the meson field. The results under (iii) are of particular interest; the author derives the classical Liénard-Wiechert potential of a moving electron and obtains the Lorentz-Dirac equations of motion for an electron.

E. T. Copson (Dundee).

Bremekamp, H. On the partial differential equations occurring in the theory of the elastic plate. Nieuw Arch. Wiskunde (2) 22, 189–199 (1946).

Let C be a closed curve in the (x, y)-plane and D the open region interior to C. Under certain restrictions on C the author proves that there exists a unique solution u of

$$\Delta\Delta u+\sum_{i=0}^{3}a_i(x, y)\frac{\partial^3 u}{\partial x^i\partial y^{3-i}}+\sum_{j=0}^{2}b_j(x, y)\frac{\partial^2 u}{\partial x^j\partial y^{2-j}}+\sum_{k=0}^{1}C_k(x, y)\frac{\partial u}{\partial x^k\partial y^{1-k}}+fu=0$$

in D, such that u and $\partial u/\partial n$ assume given values on the curve C. *F. G. Dressel* (Durham, N. C.).

Difference Equations, Special Functional Equations

Benscoter, S. U. Orthogonal functions used in the solution of linear difference problems. J. Appl. Mech. 13, A-281–A-283 (1946).

The solution of the difference equation, known as the equation of three moments in continuous beam theory,

$$\gamma m_{x-1}+(\varphi_a+\varphi_b)m_x+\gamma m_{x+1}=-\delta_x, \quad x=1, 2, 3,$$

in which γ, φ_a, φ_b are constants, is given in the form $[m]=-[T][\Lambda]^{-1}[T][\delta]$, in which $[m]$ and $[\delta]$ are column matrices with elements m_x, δ_x, $x=1, 2, 3$; $[\Lambda]^{-1}$ is a diagonal matrix with element in the nth row and nth column given by the reciprocal of $\lambda_n=\varphi_a+\varphi_b+2\gamma\cos\frac{1}{4}n\pi$, $n=1, 2, 3$; $[T]$ is a square orthogonal matrix with the element in ith row and jth column given by $2^{-\frac{1}{2}}\sin\frac{1}{4}ij\pi$, $i, j=1, 2, 3$.

D. Moskovitz (Pittsburgh, Pa.).

Wright, E. M. The non-linear difference-differential equation. Quart. J. Math., Oxford Ser. 17, 245–252 (1946).

The author considers the asymptotic expansion for exponentially small solutions of the nonlinear difference-differential equation (1) $\Lambda_1\{y(x)\}+\Lambda_2\{y(x)\}=v(x)$, where

$$\Lambda_1\{y(x)\}=y^{(n)}(x)+\sum_{\mu=1}^{m}\sum_{\nu=0}^{n-1}a_{\mu\nu}y^{(\nu)}(x+b_\mu),$$

$$\Lambda_2\{y(x)\}=\sum_{\lambda}A_\lambda y^{(\beta_{\lambda,1})}(x+b_{\lambda,1})y^{(\beta_{\lambda,2})}(x+b_{\lambda,2}),\cdots,$$

and $v(x)$ is a known function of the real variable x. The numbers a, b, A are independent of x and the b are real; $m\geqq1$, $n\geqq1$, $0\leqq\beta_{\lambda,i}\leqq n$; $\Lambda_2\{y(x)\}$ has a finite number of terms and each term has at least two y-function factors. The associated transcendental equation

$$\tau(s)=s^n+\sum_{\mu=1}^{m}\sum_{\nu=0}^{n-1}a_{\mu\nu}s^\nu e^{b_\mu s}=0 \tag{2}$$

plays a role in the development. The main result is the following theorem. If (i) $y(x)$ is a solution of (1); (ii) $v(x)$ and $y^{(\nu)}(x)$ are continuous for $x>K$ and of bounded variation in any finite interval (K, X), where $X>K$; (iii) $v(x)=O(e^{-(C-\epsilon)x})$, $y^{(\nu)}(x)=O(e^{-(c-\epsilon)x})$ for $0<c<C$ and any $\epsilon>0$; then there are polynomials $P_r(x)$, $1\leqq r\leqq R$, $Q_r(x)$, $1\leqq r\leqq R'$, and a sum $\eta(x)=\sum_{r=1}^{R}P_r(x)e^{s_rx}+\sum_{r=1}^{R'}Q_r(x)e^{S_rx}$ such that $y^{(\nu)}(x)=\eta^{(\nu)}(x)+O(e^{-(C-\epsilon)x})$. The degree of the polynomial $P_r(x)$ is at least one less than the order of the zero $s=s_r$ of (2). The numbers R and R' are related to the number of zeros of (2) whose real parts lie between $-C$ and $-c$.

D. Moskovitz (Pittsburgh, Pa.).

Bradley, F. W. Some functional equations. Proc. Math. Phys. Soc. Egypt 3 (1945), 49–52 (1946). (English. Arabic summary)

This is an expository paper on functional equations occurring in the theory of commutable functions of a real variable. Cf. papers by A. G. Walker and by Batty and Walker [Quart. J. Math., Oxford Ser. 17, 65–82, 83–92, 145–152; these Rev. 8, 19, 199].

E. H. Rothe (Ann Arbor, Mich.).

***Badescu, Radu. Sopra una generalizzazione dell'equazione funzionale di Poinsot.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 154–161. Edizioni Cremonense, Rome, 1942.

The equation is $f(z)-\lambda f(\alpha z)=p(z)$.

Integral Equations

Bădescu, Radu. Sur une extension des théorèmes de Fredholm. Ann. Sci. Univ. Jassy. Sect. I. 27, 78–85 (1941).

It is shown that for a class of kernels $K(z, s)$ the three classical theorems of Fredholm hold in the complex plane for the equation

$$\varphi(z)-\lambda\int_C K(z, s)\varphi(s)ds=\psi(z), \tag{1}$$

where C is an analytic closed curve bounding a simply connected domain D and $\psi(z)$ is analytic in $D+C$. The study of (1) is related to that of a functional equation $\varphi(z)-\lambda\varphi(\theta(z))=\psi(z)$, where $\theta(z)$ is a suitable analytic function.

W. J. Trjitzinsky (Urbana, Ill.).

Zaanen, A. C. On the theory of linear integral equations. V. Nederl. Akad. Wetensch., Proc. 49, 571–585 = Indagationes Math. 8, 352–366 (1946).

The author applies the results of paper I of this series [same Proc. 49, 194–204 = Indagationes Math. 8, 91–101 (1946); these Rev. 8, 28] to obtain more refined results than those in papers IV and IVa [same Proc. 49, 409–416, 417–423 = Indagationes Math. 8, 264–271, 271–278 (1946); these Rev. 8, 211] for kernels $K(x, y)$ of the form $A(x)H(x, y)$, where $H(x, y)$ is a positive definite Hermitian L^2 kernel and $A(x)$ is a real bounded measurable function. The expansion theorems assume a simpler form, and in certain expansions some awkward additional terms, necessary in the general case, can be dropped. When $H(x, y)$ is continuous, a generalization of Mercer's theorem on the uniform convergence of the expansion of a positive definite kernel in terms of its eigen-functions is proved. The paper concludes with some examples showing how far the results obtained are best possible.

F. Smithies.

Zaanen, A. C. On the theory of linear integral equations. VI. Nederl. Akad. Wetensch., Proc. 49, 608–621 = Indagationes Math. 8, 367–380 (1946).

The results obtained in this paper are broadly similar in character to those of paper V of the series [see the preceding review]. The kernels considered are those of the form $K(x, y)=\int A(x, z)H(z, y)dz$, where A and H are Hermitian L^2 kernels and H is positive definite. Results on the uniform convergence of the expansions are obtained when A and H are required to satisfy a variety of continuity conditions.

F. Smithies (Cambridge, England).

Ḩarazovi, D. On linear integral equations whose kernels are integral rational functions of a parameter. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 6, 663–669 (1945). (Georgian and Russian)

The author studies the equation

$$u(x)-\int_a^b G(x, y; \lambda)u(y)dy=0, \tag{1}$$

where $G(x, y; \lambda)=\sum_0^m G_n(x, y)\lambda^n$ (λ complex; G_n real symmetric, L_2 in (x, y) on $a\leqq x, y\leqq b$). A number of theorems are stated without proof. Some of the typical results are as follows. If m is odd and for all φ satisfying (2) $\int\varphi^2dx=1$ one has

$$T(\varphi)=\int\int G_0(x, y)\varphi(x)\varphi(y)dxdy<1, \tag{3}$$

then (I) there exists at least one real characteristic value of the kernel G and (II) the set of characteristic values has no finite limit points. If $m(>2)$ is even and (3) holds and G_m is not semidefinite negative, then (I), (II) can be asserted. If the G_n $(n=2, \cdots, m)$ are semidefinite positive and $\int\varphi^2dx-T(\varphi)>0$ (when (2) holds) then (I), (II) will hold and there exist no nonreal characteristic values λ such that $0<\arg\lambda\leqq\pi/(m-1)$, $2\pi-\pi/(m-1)\leqq\arg\lambda<2\pi$.

W. J. Trjitzinsky (Urbana, Ill.).

Vekua, N. Singular integral equations of general form with discontinuous coefficients. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **6**, 3–10 (1945). (Georgian. Russian summary)

The equation studied is

$$(1)\qquad A(t_0)\varphi(t_0)+(\pi i)^{-1}\int_L K(t_0, t)(t-t_0)^{-1}\varphi(t)dt=f(t_0),$$

where L is a "smooth" simple closed contour; A, K, f are assigned on L, have discontinuities of the first kind at $a_1, \cdots, a_s$ and satisfy a Lipschitz condition on each arc free of the a_j (in each variable separately); integrations are in the sense of Cauchy principal values. Solutions $\varphi(t_0)$ are sought of Hölder class on L, with possible discontinuities at the a_j of order less than unity. The author extends to (1) the developments given by him in a more special case [same Bull. **5**, 125–134 (1944); these Rev. **7**, 207].

W. J. Trjitzinsky (Urbana, Ill.).

Vekua, N. Systems of singular integral equations of general form with discontinuous coefficients. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **6**, 185–194 (1945). (Georgian. Russian summary)

The author studies a system of equations of the form

$$(1)\qquad A(t_0)\varphi(t_0)+(\pi i)^{-1}\int_L K(t_0, t)(t-t_0)^{-1}\varphi(t)dt=f(t_0),$$

where L is a "smooth" simple closed curve; A, K are assigned matrices of n^2 elements, with discontinuities of the first kind at points $a_1, \cdots, a_s$ on L, satisfying a Lipschitz condition on each arc free of the a_j; f is an n-dimensional vector of Hölder class on L except possibly at the a_j, where there may be discontinuities of the first kind; $\varphi(t_0)$ is the unknown vector, which is to be of Hölder class on every arc free of the a_j, with possible discontinuities at the a_j of order less than unity. The author extends to the system (1) the results of his work quoted in the preceding review.

W. J. Trjitzinsky (Urbana, Ill.).

Parodi, Maurice. Image de $f(\sqrt{t})$; application à la résolution d'une équation intégrale. C. R. Acad. Sci. Paris **224**, 91–92 (1947).

***Nardini, Renato. Sulla risoluzione di un'equazione funzionale del tipo misto.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 193–197. Edizioni Cremonense, Rome, 1942.

The equation is

$$\psi(x, y)=f(x, y)+\lambda\int_0^b K(x, y, z)\psi(x, z)dz+A[x, y, \psi(t, z)],$$

where A is a certain functional. Cf. Ann. Scuola Norm. Super. Pisa (2) **9**, 201–213 (1940); these Rev. **3**, 152.

Functional Analysis

Dunford, Nelson, and Segal, I. E. Semi-groups of operators and the Weierstrass theorem. Bull. Amer. Math. Soc. **52**, 911–914 (1946).

Les auteurs donnent en quelques lignes une démonstration de la généralisation de E. Hille [Proc. Nat. Acad. Sci. U. S. A. **28**, 175–178, 421–424 (1942); ces Rev. **4**, 13, 163] du théorème de M. H. Stone sur la représentation exponentielle des groupes à un paramètre d'opérateurs unitaires dans un espace de Hilbert. En appliquant ce résultat à l'espace de Banach des fonctions x uniformément continues et bornées d'une variable réelle, et au groupe des opérateurs "de translation" T_s (où $T_s x$ est la fonction $t\rightarrow x(t+s)$), ils obtiennent une démonstration du théorème de Weierstrass sur l'approximation d'une fonction continue par des polynomes. Dans une deuxième partie, ils déduisent du théorème de Weierstrass la généralisation qu'en a donnée M. H. Stone pour les fonctions continues sur un espace compact quelconque E [Trans. Amer. Math. Soc. **41**, 375–481 (1937)]; leur démonstration paraît au référent inutilement compliquée par l'introduction du dual de l'espace des fonctions continues (espace des mesures de Radon sur E) et l'utilisation du théorème de Hahn-Banach; un raisonnement direct conduit tout aussi rapidement au but [voir J. Dieudonné, Bull. Sci. Math. (2) **68**, 79–95 (1944), en particulier pp. 88–89; ces Rev. **7**, 111].

J. Dieudonné (São Paulo).

Nikodym, Otton Martin. Sur les tribus de sous-espaces d'un espace de Hilbert-Hermite. C. R. Acad. Sci. Paris **224**, 522–524 (1947).

This note consists in large part of descriptive material about (1) families of projections in Hilbert space which are Boolean rings under the natural modes of combination, (2) their effect on subspaces of the space in which they work, (3) the families (called "tribus") of closed linear manifolds which are their ranges. There are included remarks concerning measures on such Boolean rings and the associated "tribus."

M. H. Stone (Chicago, Ill.).

Neumark, M. A. On extremal spectral functions of a symmetric operator. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 7–9 (1946).

The author announces without proof further results about the spectral theory for closed symmetric operators in Hilbert space. The various spectral functions for such an operator H (generalizing the resolution of the identity in the self-adjoint case) constitute a convex set. Those which are extremal (in the usual sense of not lying internally on any segment in the set) are characterized here. The criterion is phrased in terms of the self-adjoint extension H' of H (of the second kind) whose resolution of the identity yields the particular spectral function of H under consideration. The criterion is found to be satisfied when H' is a finite-dimensional extension (the space in which H' works contains the space in which H works as the orthogonal complement of a finite-dimensional subspace). Applications are made to the case where H has deficiency-index (1, 1) and hence to the moment problem.

M. H. Stone (Chicago, Ill.).

Zitlanadze, E. S. On certain problems concerning eigenvalues for non-linear operators in the Hilbert space. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 307–309 (1946).

The author follows the line of research taken up in papers by Lusternik [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1939**, 257–264] and Soboleff [same C. R. (N.S.) **31**, 735–737 (1941); these Rev. **3**, 208] in which the Lusternik-Schnirelman method of treating variational problems by the use of topological (homotopy) classes of sets and of the notion of the category of a set is extended to prove the existence of infinitely many eigenvalues of certain not necessarily linear operators in Hilbert space.

Let $f(a)$, $\varphi_i(a)$ $(i=1, \cdots, n)$ be (not necessarily linear) functionals defined for the points a of the Hilbert space H.

Let $H_0 \subset H$ be the set of all a for which $\varphi_i(a)=0$ for $i=1, \cdots, n$; it is supposed that f and φ_i in a suitable ϵ-neighborhood $S(H_0, \epsilon)$ of H_0 possess Fréchet differentials $df(a, h)$, $d\varphi_i(a, h)$. These differentials, being linear in h, may be written as scalar products: $df=(L_f a, h)$ and $d\varphi_i=(L_{\varphi_i} a, h)$. The operator L_f is said to be generated by the differential df and any operator generated in this way by the differential of a function f is called symmetric. A point $a \subset H_0$ is called critical for the function $f(a)$ on H_0 if, for suitable constants $\lambda_1, \cdots, \lambda_n$, $L_f a=\sum_{i=1}^n \lambda_i L_{\varphi_i} a$. Let $[A]$ be a topological class of sets $A \subset H_0$. Let c be the greatest lower bound of the least upper bounds of the values of the functional $f(a)$ on the sets $A \subset [A]$. It is supposed that c is attained on a certain compact set $A_0 \subset [A]$. If then the operators L_f and L_{φ_i} satisfy a Lipschitz condition and if $(f=c)$ denotes the set of those $a \subset H_0$ for which $f(a)=c$, the following three theorems are stated and an outline of their proofs is given. (1) If the intersection $N=(f=c) \cap A_0$ is contained in the interior of H_0 then N contains at least one critical point of f. (2) Let S_H be the sphere $(a, a)=1$ and S_H^* the projective Hilbert space obtained from S_H in identifying opposite points. Denote by $[A_k^*]$ the aggregate of all closed subsets of S_H^* whose category is at least k and let L_f be a nonlinear homogeneous, symmetrical, totally continuous and positive operator defined on S_H. Then to every class $[A_k^*]$ corresponds an eigenelement a_k and an eigenvalue λ_k such that $L_f a_k=\lambda_k a_k$. (3) If the two eigenvalues λ_i and λ_j $(j>i)$ coincide then the compact intersection of the set $(f=\lambda_i)$ with the set of all eigenelements of L_f has a category on S_H^* at least equal to $j-i+1$. *E. H. Rothe* (Ann Arbor, Mich.).

Munroe, M. E. Absolute and unconditional convergence in Banach spaces. Duke Math. J. 13, 351–365 (1946).

Pour une fonction additive d'ensemble définie pour les intervalles contenus dans un intervalle borné de l'espace à n dimensions, et prenant ses valeurs dans un espace de Banach B, on définit les notions de fonction à variation bornée et de fonction absolument continue (par rapport à la mesure de Lebesgue) comme pour les fonctions d'ensemble à valeurs réelles, mais en général une fonction absolument continue n'est pas nécessairement à variation bornée; l'auteur montre que, pour que toute fonction absolument continue soit à variation bornée, il faut et il suffit que, dans B, toute série commutativement convergente soit aussi absolument convergente. On conjecture depuis longtemps que les seuls espaces de Banach ayant cette dernière propriété sont les espaces de dimension finie; l'auteur ne parvient pas à démontrer cette hypothèse, mais montre qu'elle serait exacte si tout espace de Banach vérifiait une autre propriété, pour l'énoncé de laquelle nous renvoyons à l'article lui-même; en relation avec cette propriété (qui se trouve vérifiée par tous les exemples d'espaces de Banach connus) il examine enfin diverses autres propriétés qui généralisent partiellement la notion d'orthogonalité dans l'espace de Hilbert. *J. Dieudonné* (São Paulo).

Bochner, Salomon, and Fan, Ky. Distributive order-preserving operations in partially ordered vector sets. Ann. of Math. (2) 48, 168–179 (1947).

If, for every real continuous function f on the unit interval, $A(f)$ is an element of a partially ordered real vector set (which need not possess a topological structure and need not form a lattice), and if A is distributive and takes nonnegative values for nonnegative functions, then there exists a finitely additive nonnegative interval function E with values in the vector set such that $A(f)=\int_0^1 f(t)dE(t)$. An analogous result holds for complex functions on the unit circle and complex vector sets in which there is defined an adjoint operation $X \rightarrow X^*$, with the property that the self-adjoint elements form a partially ordered real vector set. On the basis of these theorems it is shown, by standard methods, that the obvious generalizations to vector-valued functions of the Hausdorff moment problem (completely monotone sequences) and the trigonometric moment problem (positive definite sequences) are valid.

P. R. Halmos (Chicago, Ill.).

Nakano, Hidegorô. Riesz-Fischerscher Satz im normierten teilweise geordneten Modul. Proc. Imp. Acad. Tokyo 18, 350–353 (1942). [MF 14767]

An abstract lattice-generalization is established for the Riesz-Fischer theorem in L_p space. Let $\mathfrak{M}$ be a conditionally σ-complete vector lattice with absolute value $|f|$ and metric $\|f\|$. It is supposed that (i) $|f| \leqq |g|$ implies $\|f\| \leqq \|g\|$ and (ii) $f_n \rightarrow 0$ in order convergence as $n \rightarrow \infty$ implies $\|f_n\| \rightarrow 0$ as $n \rightarrow \infty$. Completeness of $\mathfrak{M}$ with respect to the metric is not assumed. The following theorems are proved. (1) If a sequence $x_1, x_2, \cdots$ has a subsequence which is bounded in the order sense and $\|x_n - x_m\| \rightarrow 0$ as $n, m \rightarrow \infty$, then there exists an x in $\mathfrak{M}$ such that $\|x - x_n\| \rightarrow 0$ as $n \rightarrow \infty$ and a suitable subsequence of the x_n converges to x in the order sense. (2) If every sequence $0 \leqq x_1 \leqq x_2 \leqq \cdots$, with $\|x_n\|$ bounded, is bounded in the order sense, then $\mathfrak{M}$ is complete with respect to the metric. The proofs make use of the projection operator in $\mathfrak{M}$,

$$[c]x = \sup\{(x \cup 0) \cap nc;\ n=1, 2, \cdots\} + \inf\{(x \cap 0) \cup (-nc);\ n=1, 2, \cdots\},$$

defined for c in $\mathfrak{M}$, $c \geqq 0$. *I. Halperin* (Kingston, Ont.).

Nakano, Hidegorô. Über Erweiterungen von allgemein teilweisegeordneten Moduln. I. Proc. Imp. Acad. Tokyo 18, 626–630 (1942). [MF 14789]

Nakano, Hidegorô. Über Erweiterungen von allgemein teilweisegeordneten Moduln. II. Proc. Imp. Acad. Tokyo 19, 138–143 (1943). [MF 14805]

Let $\mathfrak{A}$ be a partially ordered vector space, not necessarily a lattice, in which every element is bounded above by positive elements. It is furthermore assumed that (I) whenever $0 \leqq x \leqq a+b$, $a>0$, $b>0$, there exist a_1, b_1, with $0 \leqq a_1 \leqq a$, $0 \leqq b_1 \leqq b$, $x=a_1+b_1$, and (II) for every $a>0$ there is a positive functional P on $\mathfrak{A}$ with $P(a)>0$. The author constructs a partially ordered vector space $\mathfrak{B}$ and a linear, $<$-preserving mapping of $\mathfrak{A}$ on a subspace $\mathfrak{B}_1$ of $\mathfrak{B}$ such that every bounded linear functional on $\mathfrak{A}$ extends to a unique continuous functional on $\mathfrak{B}$. The mapping preserves the relations $a \cup b=c$, $a \cap b=c$ insofar as they exist in $\mathfrak{A}$. The construction is obtained by defining $\hat{\mathfrak{A}}$ as the vector lattice of all bounded linear functionals on $\mathfrak{A}$, and $\mathfrak{B}$ as the conjugate vector space of all continuous linear functionals on $\hat{\mathfrak{A}}$. This construction is used to show that an arbitrary Boolean algebra A can be imbedded in a continuous Boolean algebra B so that every nonnegative finitely-additive measure on A extends to a unique totally-additive measure on B.

Sufficiency conditions are obtained which ensure that $\mathfrak{B}_1=\mathfrak{B}$, for example, if $\mathfrak{A}$ satisfies the conditions of theorem 2 in the preceding review. Conditions are found which ensure that a given positive linear functional on a vector lattice $\mathfrak{N}$ extends to a unique continuous functional on a restricted partially ordered vector space $\mathfrak{M}$ containing $\mathfrak{N}$. These con-

ditions are shown to hold whenever $\mathfrak{N}$ consists of all continuous functions and $\mathfrak{M}$ of all bounded Baire functions on a bicompact space. *I. Halperin* (Kingston, Ont.).

Kalisch, G. K. On p-adic Hilbert spaces. Ann. of Math. (2) 48, 180–192 (1947).

Let Φ be a field with a non-Archimedean valuation $|\cdots|$ such that $|2|=1$, every element α of Φ with $|\alpha|=1$ has a square root in Φ and, as a metric space, Φ is complete and separable. A p-adic Hilbert space is a set S satisfying: (I) S is a linear space over Φ; (II) on S there is defined a symmetric bilinear inner product (a, b) with values in Φ; (III) S has a real-valued norm $|\cdots|$ such that, if $a, b\epsilon S$, $\alpha\epsilon\Phi$, then $|a|>0$ for $a\neq 0$, $|\alpha a|=|\alpha|\cdot|a|$, $|a+b|\leqq\max(|a|, |b|)$, $|(a, b)|\leqq|a|\cdot|b|$; (IV) for every $a\epsilon S$ there exists $\alpha\epsilon\Phi$ such that $|a|=|\alpha|$; (V) for every $a\epsilon S$ there exists $\bar{a}\epsilon S$ such that $\bar{a}\neq 0$, $(a, \bar{a})=|a|\cdot|\bar{a}|$; (VI) with respect to the norm S is separable and complete. Examples are the spaces of sequences $a=(\alpha_1, \alpha_2, \cdots, \alpha_n)$ or of infinite sequences $a=(\alpha_1, \alpha_2, \cdots)$ with $\lim \alpha_i=0$, $\alpha_i\epsilon\Phi$, where $|a|$ is defined as $\max_i |\alpha_i|$, and if $b=(\beta_1, \cdots)$, then (a, b) is defined as $\sum_i\alpha_i\beta_i$. Various theorems analogous to those in ordinary Hilbert space are proved about S. Thus there is a one-to-one correspondence between certain submanifolds M of S and certain linear operators P; M is the range of P and P represents an orthogonal projection on M. By a modified orthogonalization process, the existence is shown of a complete proper orthonormal set $g_1, g_2, \cdots$, i.e., a set such that $(g_i, g_j)=\delta_{ij}$, $|g_i|=1$, and for every $a\epsilon S$, $a=\sum_i(a, g_i)g_i$. By means of such a set, S is shown to be isomorphic to a space of sequences. A spectral decomposition theorem is proved for certain types of linear operators.

I. S. Cohen (Philadelphia, Pa.).

Calculus of Variations

Bieri, Hans. Beitrag zur Lösung eines Randwertproblems der Variationsrechnung. Comment. Math. Helv. 19, 227–235 (1947).

The paper is concerned with the problem of minimizing the integrals $\int\varphi^{\frac{1}{2}}ds$ in (x, y)-space and in (x, y, z)-space, where φ is of one of the forms $\varphi=ax^2+by^2+c$, $\varphi=ax^2+4by$. The extremal family is obtained together with the determinant giving the conjugate points. A discussion of the envelope of the extremals through the initial point is given for the three dimensional case. *M. R. Hestenes.*

Sólyi, Anton. Das Haarsche Lemma in der Variationstheorie und seine Anwendungen. Mat. Fiz. Lapok 48, 285–311 (1941). (Hungarian. German summary)

The well-known lemma of Haar is derived in a simplified manner, essentially by using approximations by integral means. The method applies in the n-dimensional case directly. The converse of Haar's lemma is derived in a similar manner. The second variation is shown to be accessible to a simplified treatment. For n-dimensional regular problems with an analytic integrand it is shown that, if z is an extremal that satisfies a Lipschitz condition, then the partial derivatives of the second order exist almost everywhere and are summable with their squares. For the special case of quadratic variation problems the method yields the analytic character of the extremal. The paper includes a discussion of relationships of the methods and results to previous literature. *T. Radó* (Columbus, Ohio).

***Cinquini, Silvio. Sopra una nuova estensione dei moderni metodi del calcolo delle variazioni.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 129–132. Edizioni Cremonense, Rome, 1942.

The author shows how Tonelli's direct method for establishing the existence of the extremum for integrals of the calculus of variations in the ordinary form

$$\int_a^b f(x, y(x), y'(x))dx$$

can be extended to the case in which the interval (a, b) is infinite. *From the author's summary.*

Cinquini, S. Sopra l'esistenza dell'estremo nei problemi variazionali in forma parametrica di ordine superiore. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 500–505 (1946).

This note states existence theorems (without proof) for the case of bounded regions and of unbounded regions. The integrand is supposed to be a function $F(x, y, x', y', \theta')$, where θ' is the curvature. A criterion is given for the case of unbounded regions under which the requirement that admissible curves have a point in common with a fixed bounded set is replaced by the requirement that the lengths of admissible curves have a positive lower bound.

L. M. Graves (Chicago, Ill.).

Cinquini, S. Sopra la semicontinuità degli integrali dei problemi variazionali in forma parametrica di ordine superiore. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 586–591 (1946).

This note states (without proof) necessary conditions and sufficient conditions for semicontinuity.

L. M. Graves (Chicago, Ill.).

Cinquini, Silvio. L'estremo assoluto degli integrali doppi dipendenti dalle derivate di ordine superiore. Ann. Scuola Norm. Super. Pisa (2) 10, 215–248 (1941).

The double integrals which the author considers have integrands $F(x, y, z, p, q, r, s, t)$, where r, s and t denote as usual the second derivatives of the function $z(x, y)$. The domain of integration is a fixed bounded open set D of the (x, y)-plane. Two types of convergence of sequences (z_n) are considered: type (A), in which the function z_n and its partial derivatives p_n and q_n converge uniformly on the domain D; type (B), in which the convergence is uniform on each closed set E contained in D. The first existence theorem is proved for classes of admissible functions z which are closed under type (B) convergence and which are bounded and have their partial derivatives p and q equicontinuous in the interior of D. The order of increase at infinity of the integrand F is supposed to be greater than one with respect to r, s and t, and greater than or equal to one with respect to p and q. It is next shown that the restriction that admissible functions have their partial derivatives equicontinuous may be omitted in case F has order of increase at infinity with respect to r, s and t greater than or equal to α, where $\alpha>2$. The restriction on the dependence of F on p and q may be omitted for domains D of special type, but an example shows that it cannot be omitted in general. When a restriction on the rate of increase of F with respect to z is inserted, the boundedness condition on admissible functions z may be omitted. Closure of the class of admissible functions under type (A) convergence may replace closure under type (B) convergence when the other assumptions are tightened

slightly in a variety of ways. The paper closes with some theorems on the case when the domain D is unbounded.
L. M. Graves (Chicago, Ill.).

✱Amerio, Luigi. Su alcune questioni di calcolo delle variazioni relative agli integrali doppi. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 186–192. Edizioni Cremonense, Rome, 1942.

Announcement of results published in Ann. Scuola Norm. Super. Pisa (2) **10**, 57–89 (1941); these Rev. **3**, 249.

Cinquini-Cibrario, Maria. Proprietà degli integrali delle equazioni a derivate parziali del calcolo delle variazioni. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **4(73)**, 679–698 (1940).

Let D be a closed region in the (x, y)-plane. The equation of Lagrange for the integral

$$(1)\qquad I(z)=\iint_D f(x, y, z, p, q)dxdy, \quad p=\frac{\partial z}{\partial x}, \quad q=\frac{\partial z}{\partial y},$$

is

$$(2)\qquad f_z-df_p/dx-df_q/dy=0.$$

Let $z_0(x, y)$ denote a solution of (2) and $z_1(x, y)$ a function absolutely continuous in D and such that, on the boundary of D, $z_0(x, y)=z_1(x, y)$. The paper gives sufficient conditions on the function $f(x, y, z, p, q)$ such that if $I(z_1)$ exists then $I(z_0)$ will exist and furthermore $I(z_0)\leqq I(z)$. Also conditions on f are given which insure that a minimizing function $Z(x, y)$ of (1) will have finite first partial derivatives and satisfy (2) in D.
F. G. Dressel (Durham, N. C.).

Lévy, Paul. Exemples de contours pour lesquels le problème de Plateau a 3 ou $2p+1$ solutions. C. R. Acad. Sci. Paris **224**, 325–327 (1947).

The author states the following principle. Let Γ (or $MABNA'B'M$) be a simple twisted closed Jordan curve in space. If the distances $\overline{AA'}$ and $\overline{BB'}$ are sufficiently small, then there may be exactly three minimal surfaces spanning the contour Γ. This principle is used to indicate how certain explicit curves Γ, described by the author, may have exactly $2p+1$ minimal surfaces spanning the contour, for positive integral p. Some of the curves described by the present author are pictured by Courant [Amer. Math. Monthly **47**, 167–174 (1940), especially figs. 7, 8, 9 on p. 171; these Rev. **1**, 270].
M. O. Reade (Ann Arbor, Mich.).

Theory of Probability

Bose, R. C. The patch number problem. Science and Culture **12**, 199–200 (1946).

A square is divided into n^2 small squares by lines parallel to the sides. The small squares are divided at random into black and white squares. Let p (g) be the probability that a square is black (white). A black patch is an array of black squares such that it is possible to walk from one square of the array to any other through black territory. A white patch is similarly defined but contact at corners only is also permissible. Let X be the difference between the number of black patches and the number of white patches embedded in black patches. The author states without proof that $E(X)=p+2(n-1)pg(n-1)^2(pg^2-g^2p)$.
H. B. Mann (Columbus, Ohio).

Bronowski, J., and Neyman, J. The variance of the measure of a two-dimensional random set. Ann. Math. Statistics **16**, 330–341 (1945). [MF 15466]

Let R be a fixed rectangle with sides a and b and R' a concentric parallel rectangle with sides $a+c$, $b+c$ ($c>0$). Consider a number N of rectangles of fixed dimension and parallel to R such that (1) the number N is a random variable with known probability distribution, (2) the probability that k out of the N rectangles have their centers within a prescribed subdomain D of R' is given by a binomial distribution with $p=|D|/|R|$. Let X be the measure of the subset of R' covered by the variable rectangles. The authors compute the first two moments of X. This result is a sequel to one of Robbins [same Ann. **15**, 70–74 (1944); these Rev. **6**, 5].
W. Feller (Ithaca, N. Y.).

Robbins, H. E. On the measure of a random set. II. Ann. Math. Statistics **16**, 342–347 (1945). [MF 15467]

[Part I is the paper quoted in the preceding review.] The problem of the paper reviewed above is solved for n dimensions. The analogous plane problem for circles is also solved.
W. Feller (Ithaca, N. Y.).

Garwood, F. The variance of the overlap of geometrical figures with reference to a bombing problem. Biometrika **34**, 1–17 (1947).

Congruent plane figures of known shape are placed so that their positions are statistically independent and underlying the same known probability distribution. The number of such figures is a random variable. Let A be a fixed domain and f the fraction of area of A not covered by any of the figures. A theorem of Robbins [Ann. Math. Statistics **15**, 70–74 (1944); these Rev. **6**, 5] shows a way to compute the mean and variance of f. The author investigates in detail a few cases such as covering a rectangle by circles with centers in a concentric "rectangle with rounded corners." The number of circles follows a Poisson distribution. Simple numerical approximations are given. The theory is compared with laboratory experiments. In a footnote added in proofs it is stated that part of the results are obtained in the papers of Bronowski and Neyman and Robbins reviewed above.
W. Feller (Ithaca, N. Y.).

Santaló, L. A. On the first two moments of the measure of a random set. Ann. Math. Statistics **18**, 37–49 (1947).

Let R and R^1 be the rectangles $0<x<A$, $0<y<B$ and $-\delta<x<A+\delta$, $-\delta<y<B+\delta$, respectively. Consider N congruent rectangles of dimensions $a\times b$ (with $(a^2+b^2)\leqq\min(A^2, B^2, \delta^2)$) chosen independently and at random so that their centers have uniform probability density in R^1 and that all directions are equally probable. Let S be the union of the N rectangles. The author finds the expectation and the variance of the area of SR. A similar problem is solved for n dimensions with the N rectangles replaced by spheres. The derivation follows the method of Robbins [cf. the second preceding review].
W. Feller (Ithaca, N. Y.).

Törnqvist, Leo. On the distribution function for a function of n statistic variables and the central limit theorem in the mathematical theory of probability. Skand. Aktuarietidskr. **29**, 206–229 (1946).

Investigations concerning the central limit theorem are usually restricted to sums of mutually independent random variables. The author generalizes the problem in two directions. For every $n\geqq1$ let there be given an n-dimensional random vector $X=(X_1, \cdots, X_n)$, where the variables are

not necessarily independent. Let $u_n(X)$ be a given sequence of functions of $(X_1, \cdots, X_n)$. The sequence of random variables $\{u_n(X)\}$ is said to belong to the normal domain if there exist sequences of numbers $\{a_n\}$ and $\{b_n\}$ such that the distribution function $W_n(a_nt+b_n)=\Pr\{u_n(X)\leqq a_nt+b_n\}$ tends to the normalized Gaussian distribution $\Phi(t)$. If $F_n(x)$ is the n-dimensional distribution of the vector X, and $H(t)=\frac{1}{2}\pm\frac{1}{2}$ according as $t>0$ or $t<0$, then

$$W_n(a_nt+b_n)=\int_{-\infty}^{\infty}H(a_nt+b_n-u_n(x))dF_n(x).$$

Using the expansion

$$H(t-(u_n(x)-b)/a)=\sum_{-\infty}^{\infty}c_k\exp\{2\pi ki\Phi(t)\}$$

the author shows that a function $b=b(a)$ exists for which

$$(1)\qquad w_{n,a}(t)\equiv W_n(at+b)-\Phi(t)=\sum\gamma_k\exp\{2\pi ki\Phi(t)\},$$

where the coefficients γ_k depend on u_n and a. If $\{u_n\}$ belongs to the normal domain then one may put $b_n=b(a_n)$. The problem is now reduced to that of finding conditions under which (1) tends to zero. It is shown that the second moment

$$w_n^2(a)=\int_{-\infty}^{\infty}w_{n,a}^2(t)d\Phi(t)$$

has, as a function of a, a maximum equal to 1/12. The sequence $\{u_n\}$ belongs to the normal domain if $\{a_n\}$ can be determined so that $w_n^2(a_n)\to 0$. It is necessary and sufficient that the individual coefficients in (1) tend to zero. Various sufficient conditions are given but they are too long to quote here. *W. Feller* (Ithaca, N. Y.).

Dubourdieu, J. Sur une généralisation d'un théorème de M. B. de Finetti et son application à la théorie collective du risque. C. R. Acad. Sci. Paris **224**, 514–516 (1947).

Let $R(x)$ be a nondecreasing function. Consider a game where the stakes are adjusted from trial to trial so that, if x is the player's fortune before the trial and X the gain, $E(\exp\{R(x)-R(x+X)\})=1$. By a simple argument the author shows that the probability of ruin does not exceed $\exp\{R(0)-R(a)\}$, where a is the initial capital of the player. Using this theorem he gives a simple proof and generalization of a theorem of I. Laurin in the collective risk-theory [Skand. Aktuarietidskr. **13**, 84–111 (1930)]. *W. Feller* (Ithaca, N. Y.).

Campagne, C. The theorem of Hattendorf and its general validity by the theorem of Cantelli. Verzekerings-Arch. **24**, 121–144 (1943). (Dutch)

Cantelli, F. P. Osservazioni sulla formula di Hattendorff. Giorn. Ist. Ital. Attuari **11**, 261–269 (1940). [MF 16618]

The author has previously [Atti Soc. Ital. Progresso Sci. **18**$_{II}$, 25–30 (1929)] pointed out that Hattendorff's formula for the variance of a risk is really a direct consequence of the additivity of the variance of dependent uncorrelated random variables. He now shows various applications of the method by deriving several actuarial formulae, avoiding the usual unnecessary complications. *W. Feller.*

Invrea, Raffaele. Il rischio medio di un'operazione assicurativa e l'applicazione di un teorema del Cantelli. Giorn. Ist. Ital. Attuari **12**, 167–190 (1941).

Using Cantelli's remarks [cf. the preceding review] the author derives an explicit formula containing as special cases Hattendorff's formula and several of its known generalizations. *W. Feller* (Ithaca, N. Y.).

Invrea, R. Ancora a proposito di rischio medio. Giorn. Ist. Ital. Attuari **13**, 54–56 (1942).

Simplifications to the derivation of the formula mentioned in the preceding review. *W. Feller.*

de Finetti, B. Il problema dei "Pieni." Giorn. Ist. Ital. Attuari **11**, 1–88 (1940). [MF 16623]

A detailed and technical study of various problems of the risk theory for insurance companies. The main problem is that of optimal methods of reinsurance. Attention is paid to the statistical interdependence of the individual policies, the influence of the reserve-size on the future developments, etc. Asymptotical formulae are derived which in part coincide with those of the collective risk theory. In discussing the latter the author claims that it introduces unnecessary restrictions and an undesirable formal analytical apparatus. *W. Feller* (Ithaca, N. Y.).

Ottaviani, G. La teoria del rischio del Lundberg e il suo legame con la teoria classica del rischio. Giorn. Ist. Ital. Attuari **11**, 163–189 (1940). [MF 16620]

Continuing de Finetti's remarks [cf. the preceding review], the author subjects the collective risk theory to a critical analysis. He discovers that the theory is actually consistent with classical ideas which it combines with modern limiting processes. This is not a new idea. [This paper has been discussed by Segerdahl, Skand. Aktuarietidskr. **25**, 43–83 (1942); these Rev. **8**, 215.] *W. Feller* (Ithaca, N. Y.).

***Cramér, Harald. On the theory of stochastic processes.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 28–39. Jul. Gjellerups Forlag, Copenhagen, 1947.

Expository paper. *J. L. Doob* (Urbana, Ill.).

***Cramér, Harald. Lundberg's risk theory and the theory of stochastic processes.** Försäkringsmatematiska Studier Tillägnade Filip Lundberg, pp. 25–31. Stockholm, 1946. (Swedish)

The author emphasizes that the collective risk theory [which was started by F. Lundberg about 1903] should be regarded as part and predecessor of the modern theory of discontinuous Markov processes [cf. the following review]. *W. Feller* (Ithaca, N. Y.).

***Davidson, Åke. On the problem of ruin in the collective risk theory under the assumption of variable safety loading.** Försäkringsmatematiska Studier Tillägnade Filip Lundberg, pp. 32–47. Stockholm, 1946. (Swedish)

Underlying the collective theory of risk is a Markov process in which a random variable X (the reserve) changes in a discontinuous manner. In any time interval dt the probability of a jump is $dt+o(dt)$ and the magnitude of the jump is a random variable Y with given probability distribution and $E(Y)=1$; with probability $1-dt+o(dt)$ no jump occurs and then X changes by an amount $(1+\lambda(X))dt$. The function $\lambda(X)$ is the safety loading. The theory of ruin has been developed for the case $\lambda=$constant [cf. C. O. Segerdahl, On Homogeneous Random Processes and Collective Risk Theory, Stockholm thesis, 132 pp., Uppsala, 1939]. Some of the results are generalized to the case of arbitrary functions $\lambda(X)$. *W. Feller* (Ithaca, N. Y.).

***Simonsen, W. On the foundation of the collective risk theory.** Försäkringsmatematiska Studier Tillägnade Filip Lundberg, pp. 246–264. Stockholm, 1946. (Danish)

The classical theory of insurance studies the risk incurred in individual contracts separately and computes the total risk by an averaging process. The collective theory of risk [cf. the preceding review] considers only the total risk as the sample function of a compound Poisson process. The author shows how the classical theory leads, in a natural way, to the fundamental concepts of the collective theory.

W. Feller (Ithaca, N. Y.).

***Elfving, G. On compound binomial processes.** Försäkringsmatematiska Studier Tillägnade Filip Lundberg, pp. 48–78. Stockholm, 1946.

The paper is devoted to Markov processes of the "pure birth" type, i.e., the variable $\mathbf{n}(t)$ of the process is integral valued and at points of discontinuity $\mathbf{n}(t+)-\mathbf{n}(t-)=1$. Let $M<N$ be two integers and $S<T$. If $\mathbf{n}(t)$ increases in $S<t<T$ from M to N and the joint distribution density of the saltus points $t_1, \cdots, t_{M-N}$ is constant in the entire region $S<t_1<\cdots<t_{M-N}<T$, the process is called binomial. Its transition probabilities are

$$(1)\qquad P_{mn}(S,t)=\binom{N-m}{n-m}\left(\frac{t-S}{T-S}\right)^{n-m}\left(\frac{T-t}{T-S}\right)^{N-n},$$

its absolute probabilities $P_n(t)=P_{M,n}(S,t)$. Any process with the transition probabilities (1) is called a generalized binomial process. If $N\to\infty$, $T\to\infty$, $N/T\to\infty$, the process converges to the familiar Poisson process. Let $\mathbf{n}(t)$ be the variable of an arbitrary process. The conditions $\mathbf{n}(\theta_1)=\lambda_1$, $\mathbf{n}(\theta_2)=\lambda_2$ define in $\theta_1<t<\theta_2$ a new process called "interval subprocess." If all such subprocesses are binomial, $\mathbf{n}(t)$ defines a compound binomial or B-process. Its transition probabilities are shown to be of the form

$$(2)\qquad P_{mn}(s,t)=\frac{C_n(t)}{C_m(s)}\,\frac{(t-s)^{n-m}}{(n-m)!}$$

and the coefficients of Kolmogoroff's differential equations are expressed in terms of $C_n(t)$. In every finite interval (S,T) a B-process can be represented as a weighted mean of generalized binomial processes and the statement holds also for $T=\infty$ with the Poisson process playing the role of the binomial. The distribution of $\mathbf{n}(t)/t$ tends, as $t\to\infty$, to the weight function $U(x)$. [Cf. the following review.]

W. Feller (Ithaca, N. Y.).

Elfving, G. Contributions to the theory of integer-valued Markoff processes. Skand. Aktuarietidskr. 29, 175–205 (1946).

Consider a Markov process with denumerably many states whose transition probabilities depend on a parameter x (e.g., the Poisson process). Integrating with respect to an arbitrary weight function $U(x)$ we obtain a new or "compound" process which in general is not of the Markov type. The author shows that compounding with an arbitrary $U(x)$ will lead to a Markov process if, and only if, either the transition probabilities $P_{mn}(s,t)$ or the inverse probabilities $Q_{m,n}(s,t)=P_m(s)P_{mn}(s,t)/P_n(t)$ are independent of x; here $P_m(t)$ stand for the so-called absolute probabilities (note that the process is not necessarily defined for all values of s and t). A large part of the paper is devoted to the general theory and the Kolmogoroff equations and to properties of subprocesses [cf. the preceding review].

W. Feller (Ithaca, N. Y.).

***Karhunen, Kari. Lineare Transformationen stationärer stochastischer Prozesse.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 320–324. Jul. Gjellerups Forlag, Copenhagen, 1947.

Let $\{y(t)\}$ be the chance variables of a stationary (wide sense) complex-valued stochastic process, $-\infty<t<\infty$. The author remarks that the spectral representation $y(t)=\int_{-\infty}^{\infty}e^{it\lambda}dZ(\lambda)$ [Cramér, Ark. Mat. Astr. Fys. 28B, no. 12 (1942); these Rev. 4, 13] leads to a simple representation of any linearly derived process; that is, if $y(t)$ is the limit in mean of sums of the form $\sum_\nu c_\nu y(t+t_\nu)$, then

$$y(t)=\int_{-\infty}^{\infty}\gamma(\lambda)e^{it\lambda}dZ(\lambda).$$

This fact was also noted by Kolmogoroff [Byull. Moskov. Gos. Univ. Matematika 2, no. 6 (1941); these Rev. 5, 101] in the discrete parameter case. The author then applies this representation to finding the solution for $x(t)$ of $\sum_0^n a_j d^j x(t)/dt^j=y(t)$ and $\sum_0^n a_j x(t+jh)=y(t)$, where the $x(t)$ and $y(t)$ processes are stationary (wide sense), by finding the corresponding functions $\gamma(\lambda)$.

J. L. Doob.

Levinson, Norman. The Wiener RMS (root mean square) error criterion in filter design and prediction. J. Math. Phys. Mass. Inst. Tech. 25, 261–278 (1947).

Let $\cdots, a_0, a_1, \cdots$ and $\cdots, b_0, b_1, \cdots$ be sequences of chance variables determining stationary stochastic processes (the "message" and the "signal," respectively). The problem of designing a filter to separate the message from the noise b_j-a_j is formulated as the problem of finding coefficients $A_0, \cdots, A_M$ which minimize the expectation of $\{a_k-\sum_{n=0}^M A_n b_{k-n}\}^2$. [The author formulates his averages as time averages, that is, sample averages, rather than ensemble averages, that is, expectations. Contrary to his statement, the two will not always be the same without a further hypothesis, say that of metric transitivity.] Here M is to be chosen so large that the minimum I_M obtained will not be too much larger than $I=\lim_{M\to\infty} I_M$. The A_j are then the solutions of the standard normal equations and can be expressed in terms of the a and b correlation and cross correlation functions. Practical suggestions are given for the design of 4 terminal RC networks (filters) which will approximate the minimizing coefficients A_n. Finally, the problem is generalized to include prediction. Wiener, using the full past, so that $M=\infty$, has found expressions for the minimizing function of the past [Bol. Soc. Mat. Mexicana 2, 37–45 (1945); these Rev. 7, 461]. [For closely related work see Kolmogoroff, C. R. Acad. Sci. Paris 208, 2043–2045 (1939) and later papers; Krein, C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 91–94, 306–309 (1945); these Rev. 7, 156, 61]; these references contain expressions for the error I, and thus contain the conditions of full predictability $I=0$.]

J. L. Doob (Urbana, Ill.).

Ferrand, Jacqueline, et Fortet, Robert. Sur des suites arithmétiques équiréparties. C. R. Acad. Sci. Paris 224, 516–518 (1947).

The authors state a variety of results which extend considerably our knowledge of the interrelations between probability theory and the theory of gap series. (I) Let $\{n_j\}$ be an increasing sequence of positive numbers such that $n_{j+1}/n_j>\rho>1$ and $\{c_j\}$ a sequence of complex numbers such that $\sum|c_j|^2=\infty$. Let P_r and Q_r be the real and imaginary

parts of

$$W_r = a_r^{-1}\sum_{j=1}^{r} \exp(2\pi i n_j x), \qquad a_r^2 = \sum_{1}^{r} |c_j|^2,$$

$0 \leq x \leq 1$. Then the joint distribution function of P_r and Q_r (i.e., the measure of the set of those x for which simultaneously $P_r < u$ and $Q_r < v$) approaches, as $r \to \infty$, the Gaussian distribution with density $\pi^{-1} \exp(-u^2 - v^2)$. [For results of this nature derived under the much stronger assumption $n_{j+1}/n_j \to \infty$, see Kac, Amer. J. Math. 61, 473–476 (1939)]. As an immediate consequence one obtains that $\sum c_j \exp(2\pi i n_j x)$ diverges almost everywhere. [In the case that n_j are integers this reduces to the classical result of Zygmund. For the extension to the nonharmonic case see Kac, Duke Math. J. 8, 541–545 (1941); these Rev. 3, 107, and Hartman, ibid. 9, 404–405 (1942); these Rev. 4, 39.]

(II) Let $f(x)$ be a periodic function (of period 1) satisfying a Lipschitz condition and such that $\int_0^1 f(x)dx = 0$. Put

$$Y_r = \sum_{j=1}^{r} f(n_j x), \quad \sigma_r^2 = r^{-1}\int_0^1 Y_r^2 dx$$

and assume that lim inf $\sigma_r > 0$. The limiting distribution of $Y_r/\sigma_r r^{\frac{1}{2}}$ need not be Gaussian and, in fact, need not exist at all. This is of particular interest because for $n_j = a^j$, a an integer greater than 1, the limiting distribution is always Gaussian [see Fortet, Studia Math. 9, 54–70 (1940); these Rev. 3, 169, and Kac, Ann. of Math. (2) 47, 33–49 (1946); these Rev. 7, 436].

(III) The limiting distribution of $Y_r/\sigma_r r^{\frac{1}{2}}$ is Gaussian if one assumes that, for all j, n_{j+1}/n_j is an integer (which may depend on j). This is a direct generalization of the case $n_j = a^j$ mentioned above. As a corollary of a stronger result (analogous to the one stated in (I)) one also obtains a theorem concerning divergence almost everywhere of $\sum c_j f(n_j x)$ when n_{j+1}/n_j is an integer. *M. Kac.*

Cameron, R. H., and Martin, W. T. The behavior of measure and measurability under change of scale in Wiener space. Bull. Amer. Math. Soc. 53, 130–137 (1947).

Let $C: \{x(t)\}$ be the space of continuous functions, $x(0)=0$, $0 \leq t \leq 1$, and suppose that a probability measure is defined on C-sets following Wiener so that $x(t)-x(s)$ for fixed s and t has a Gaussian distribution with mean 0 and variance $\frac{1}{2}|t-s|$; increments of the functions over nonoverlapping intervals are independent. The authors prove that if $0 = t_0^{(\nu)} < t_1^{(\nu)} < \cdots < t_{n_\nu}^{(\nu)} = 1$ then $\sum_{j=1}^{n_\nu} \{x(t_j^{(\nu)}) - x(t_{j-1}^{(\nu)})\}^2 \to \frac{1}{2}$ with probability 1 when $\nu \to \infty$, and $t_j = j/2^\nu$, $n_\nu = 2^\nu$. [This theorem was proved more generally by P. Lévy, Amer. J. Math. 62, 487–550 (1940); these Rev. 2, 107; he supposed only that the $(\nu+1)$th division was a subdivision of the νth and that $\max_j (t_{j+1}^{(\nu)} - t_j^{(\nu)}) \to 0$.] The authors then use this result to prove several theorems on the effect of the transformation taking $x(t)$ into $\lambda x(t)$. For example, denoting the image of the C-set E by λE, if $f(\lambda)$ is an arbitrary function satisfying $0 \leq f(\lambda) \leq 1$ for $0 < \lambda$, there is a C-set E for which λE is C-measurable for all $\lambda > 0$ and such that the measure of λE is $f(\lambda)$. *J. L. Doob* (Urbana, Ill.).

Bunimovich, V. I., and Leontovich, M. A. On the distribution of the number of large deviations in electric fluctuations. C. R. (Doklady) Acad. Sci. URSS (N.S.) 53, 21–23 (1946).

Statistical properties of the noise voltage across the condenser of an RLC circuit are studied. It is shown by an intuitive argument that if the damping is small the average number of a-values of the noise voltage $x(t)$ (normalized by $\bar{x}^2 = 1$), per unit time, is (*) $(\frac{1}{2}\omega_0/\pi)\exp(-a^2)$, where ω_0 is the proper frequency of the circuit. It should be pointed out that (*) and more general results were obtained by S. O. Rice and the reviewer [see Rice, Bell System Tech. J. 24, 46–156 (1945); these Rev. 6, 233]. In particular, the assumption that the damping is small is redundant inasmuch as (*) holds for all RLC circuits.

The authors also study the problem of a-values of the amplitude curve which they define as the "broken line joining the maxima of $x(t)$." The reviewer was, however, unable to understand the connection between this definition and the subsequent calculations. *M. Kac.*

Lovera, G. Sulle coincidenze triple accidentali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 964–969 (1946).

Suppose that each of three counters receives impulses in accordance with the Poisson law and that the three series are independent. Following each registration a counter is locked during a fixed time τ and impulses are registered only if they arrive when the counter is not locked. A triple coincidence can take place only if all three counters have made a registration within a time interval of length τ; if $x < \tau$ is the distance between the extreme registrations, there is probability $p(x)$ that a triple coincidence takes place. (In the usual theory one assumes $p(x) = 1$.) The expected number of triple coincidences is calculated in terms of the moments of $p(x)$. *W. Feller* (Ithaca, N. Y.).

Mathematical Statistics

✱Weatherburn, C. E. A First Course in Mathematical Statistics. Cambridge, at the University Press; New York, The Macmillan Company, 1946. xv+271 pp. $3.50.

This book is intended "to provide a mathematical text adapted to the needs of the student with an average mathematical equipment, including an ordinary knowledge of the integral calculus." It assembles the formal derivations of many results used in elementary statistics. The student will find it a convenient collection of them. The first two chapters are meant as a short introduction to probability and are followed by a discussion of some standard distributions. The theory of correlation is discussed in detail. An entire chapter is devoted to "standard errors of statistics." The logic of tests of significance is described briefly and many tests of significance are discussed. Applications of chi-square, the analysis of variance and the analysis of covariance each receive a chapter. Numerous exercises enhance the value of the book as a text.

This book is not in the modern manner. A test of significance is described as based on the occurrence of a very rare event. No mention is made of the idea of power, and hence of why certain very rare events are held to confirm the null hypothesis and others to contradict it. Moreover, on pages 195–196, the author speaks of "the need of supplementing the χ^2 test with others, before deciding whether a given sample is a rare one." The subsequent example shows that the author considers a sample "rare" if it falls into the critical region of any one of a number of tests. This procedure distorts the level of significance; if enough "supple-

mentary" tests are applied, the size of the critical region may be considerable and a significant sample far from "rare." Fiducial limits for a mean μ are described simply as "the limits within which μ must lie, in order that the observed sample mean should not be significant at the prescribed level of probability." This is, of course, a correct formalization of a special procedure, but no mention is made of the operational significance of the idea. The maximum likelihood method is described for a special case without mention of its significance. *J. Wolfowitz.*

✶**Cansado, Enrique. Integral de Stieltjes-Lebesgue y sus Aplicaciones a la Estadística. [Stieltjes-Lebesgue Integral and its Applications to Statistics].** Memorias de Matematica del Instituto "Jorge Juan," no. 3. Madrid, 1946. 66 pp.

Expository paper; the author considers insufficient previous treatments of the multiple Stieltjes integral evaluated as an iterated integral (that is to say, previous treatments of conditional distributions and expectations). He is apparently unaware of Kolmogoroff's definitions [Grundbegriffe der Wahrscheinlichkeitsrechnung, Ergebnisse der Math., v. 2, no. 3, Springer, Berlin, 1933, pp. 41 et seq.]. *J. L. Doob* (Urbana, Ill.).

✶**Odhnoff, W. Some studies of the characteristic functions and the semi-invariants of Pearson's frequency-functions.** Försäkringsmatematiska Studier Tillägnade Filip Lundberg, pp. 168–179. Stockholm, 1946.

The study is based on a differential equation for the characteristic function which the author deduces from the original differential equation of the Pearson curves. *W. Feller* (Ithaca, N. Y.).

Michalup, E. Über den Begriff "Exzess" in der mathematischen Statistik. Mitt. Verein. Schweiz. Versich.-Math. 46, 231–236 (1946).

The author reviews the polemical literature concerning the geometric interpretation of the excess

$$E=3\beta_4=((\mu_4/\mu_2^2)-3)/8.$$

He concludes that the problem remains open whether there is any distribution which has positive excess in the sense of the Lindeberg definition, whose summit, however, lies below the normal curve, and conversely. *A. A. Bennett.*

Chakrabarti, M. C. A note on skewness and kurtosis. Bull. Calcutta Math. Soc. 38, 133–136 (1946).

If m_i, $i=2, 3, 4$, are moments about the mean for n real quantities which are not all equal, the upper bound of m_4/m_2^2 is shown to be $n-2+(n-1)^{-1}$. The upper bound is attained, for example, when one of the n quantities is $(n-1)^{\frac{1}{2}}$ and the others are all equal to $-(n-1)^{-\frac{1}{2}}$. A similar result, max $m_3/m_2^{\frac{3}{2}}=(n-2)(n-1)^{-\frac{1}{2}}$, due to Wilkins [Ann. Math. Statistics 15, 333–335 (1944); these Rev. 6, 91] is derived by another method. *A. M. Mood* (Ames, Iowa).

✶**Hagstroem, K.-G. Un problème du calcul stochastique.** Försäkringsmatematiska Studier Tillägnade Filip Lundberg, pp. 104–127. Stockholm, 1946.

The author describes a method of approximate interpolation for moments of a distribution function when two or more moments are known. Several practical applications are given. *W. Feller* (Ithaca, N. Y.).

Knoll, F. Über Näherungsverfahren bei empirisch gegebenen Verteilungsfunktionen und damit verbundene Korrekturformeln. Deutsche Math. 7, 187–194 (1943).

Using a presumably new form of the remainder term in the Euler-MacLaurin summation formula, the author derives some graduation formulas for semi-invariants. These formulas hold when the observed grouped values are exact (not subject to sampling variations) and the original continuous distribution has a density which is (A) a step-function with one step per group, (B) a polygonal function with one segment per group, (C) a compound of interlaced parabolas, one per group, (D) a more complex case. The effects of sampling are not discussed. *J. W. Tukey.*

Kreis, H. Lineare Abhängigkeit und Äquivalenz von Punktsystemen. Mitt. Verein. Schweiz. Versich.-Math. 46, 169–186 (1946).

Starting from the notions of normalized point systems in two or three dimensions (subject to the conditions of zero mean, unit standard deviation and given correlation coefficient) the author considers pairs of triangles in the plane and of tetrahedra in space, whose vertices yield (in either space) on normalization the same correlation coefficient. Some elementary geometric observations lead to no significant results on linear dependence. *A. A. Bennett.*

✶**Wold, Herman. A comment on spurious correlation.** Försäkringsmatematiska Studier Tillägnade Filip Lundberg, pp. 278–285. Stockholm, 1946.

The following theorem is proved. Given a set of random variables, say $\xi_1, \cdots, \xi_n$, suppose that ξ_1 is not in perfect correlation with the set $\xi_2, \cdots, \xi_n$. Let b be an arbitrary constant. There then exists a variable ξ_{n+1}, not in perfect correlation with $\xi_1, \cdots, \xi_n$, so that $b_{12\cdot 34\cdots n,\, n+1}=b$. *W. Feller* (Ithaca, N. Y.).

Fréchet, Maurice. Anciens et nouveaux indices de corrélation. Leur application au calcul des retards économiques. Econometrica 15, 1–30 (1947).

A constructive analysis of measures of correlation between two random variables x and y, based on the principle that such measures should equal zero if and only if all conditional distribution functions $F_x(y)$ coincide with the marginal distribution function $F(y)$ and equal unity if and only if y is a univalent function of x. The ordinary correlation coefficient does not fulfil these desiderata. A general type of correlation index which satisfies them is obtained by combining an average over y and x of $|F_x(y)-F(y)|$ with a similar average of $|F_x(y)-\Phi_x(y)|$, where $\Phi_x(y)$ is a distribution function which jumps from 0 to 1 as y passes the median of the distribution function $F_x(y)$. Numerical illustrations are given. *H. Wold* (Uppsala).

Geppert, Maria-Pia. Über den Vergleich zweier beobachteter Häufigkeiten. Deutsche Math. 7, 553–592 (1944).

This is a compilation of some of the statistical techniques employed in analysis of four-fold contingency tables. Techniques based on both "direct" probability methods, as the author calls them, and on inverse probability concepts are discussed. Large sample, asymptotic and exact or small-sample methods are considered. No claim to originality is made for the work. *R. A. Porter* (Raleigh, N. C.).

Gildemeister, M., und van der Waerden, B. L. Die Zulässigkeit des χ^2-Kriteriums für kleine Versuchszahlen. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **95** (1943), 145–150 (1944).

Results of actual calculation of the chance that χ^2 or $\chi_1^2=(1-N^{-1})\chi^2$ exceeds the significance levels 5%, 2%, 1% and 0.27%, when calculated for a four-fold table with one fixed margin $(n_1+n_2=N)$, and when the population p's are equal, are given for 14 combinations of (n_1, n_2) with $8\leqq N\leqq 21$ and $p=0.1(0.1)0.5$. Statistical conclusions are drawn which would not find universal agreement. *J. W. Tukey.*

Wolfowitz, J. Confidence limits for the fraction of a normal population which lies between two given limits. Ann. Math. Statistics **17**, 483–488 (1946).

Given a fixed interval I, a normal distribution with mean μ and variance σ^2 assigns probability γ to I. Given a sample of N from a normal distribution it is natural to try to base a lower bound D for γ on the sample mean $\bar{x}$ and sample variance s^2. The author constructs a function $D=D_\alpha(\bar{x}, s, N)$, such that the probability of $D<\gamma$ converges to α as N increases for each fixed μ, σ^2, where $\mu\epsilon I$. This approach is uniform when σ is bounded and μ is bounded away from the ends of I. The form of the solution is that which would be obtained by making the approximation $\bar{x}=\mu$ but allowing s/σ to have a chi-squared distribution.

J. W. Tukey (Princeton, N. J.).

Anderson, T. W. The non-central Wishart distribution and certain problems of multivariate statistics. Ann. Math. Statistics **17**, 409–431 (1946).

The noncentral Wishart distribution is the distribution of the sums a_{ij} of squares and cross-products in a sample arising from (say) N normal p-variate populations with common second order moments σ_{ij} but expected values $\mu_{i\alpha}$ varying from observation to observation. In the applications, the population centers $(\mu_{1\alpha}, \cdots, \mu_{p\alpha})$ $(\alpha=1, \cdots, N)$ often lie in a linear space of dimension $t<p$. The author and M. A. Girshick have previously derived the distribution in question for $t=1, 2$ [same Ann. **15**, 345–357 (1944); these Rev. **6**, 161]. The present paper contains the following new results. (1) The characteristic function of the distribution is given. (2) The author proves that the convolution of several noncentral Wishart distributions is another distribution of the same kind. (3) A new integral representation of the frequency function (in the general case) is derived. (4) The moments of the generalized variance $|a_{ij}|$ are given in the cases $t=1, 2$. (5) Let the $n+q$ p-dimensional variables $(z_{1\alpha}, \cdots, z_{p\alpha})$, $\alpha=1, \cdots, n$, and $(y_{1\gamma}, \cdots, y_{p\gamma})$, $\gamma=1, \cdots, q$, be independently and normally distributed with common second order moments σ_{ij} and with $E(z_{i\alpha})=0$, whereas the means $E(y_{i\gamma})=\mu_{i\gamma}$ may differ from zero. Write, furthermore, $a_{ij}=\sum y_{i\gamma}y_{j\gamma}$, $b_{ij}=\sum z_{i\alpha}z_{j\alpha}$, and $W=|b_{ij}|/|a_{ij}+b_{ij}|$; W is essentially the likelihood ratio criterion for testing the so-called Wilks-Lawley hypothesis. The author gives the moments of W for the cases $t=1, 2$. Furthermore, he indicates a number of hypotheses reducible to those mentioned. *G. Elfving* (Helsingfors).

Nandi, H. K. On the power function of Studentised D^2-statistic. Bull. Calcutta Math. Soc. **38**, 79–84 (1946).

Let $X=(X_1, \cdots, X_p)$, $Y=(Y_1, \cdots, Y_p)$ be two normally distributed random vectors with the same but unknown covariance matrix and let a sample of n_1 x's and n_2 y's be taken. Then $D^2=p^{-1}\sum_{i,j=1}^{p} c^{ij}(\bar{x}_i-\bar{y}_i)(\bar{x}_j-\bar{y}_j)$, where $(c_{ij})=(c^{ij})^{-1}$ is the sample covariance matrix and $\bar{x}_i$, $\bar{y}_i$, $i=1, \cdots, p$, are the sample means. The region $D^2\geqq D_0^2$ is used to test the hypothesis $E(x_i)=E(y_i)$, $i=1, \cdots, p$. The author proves that this region is unbiased and that it possesses certain optimum properties. These include a property previously obtained by Simaika [Biometrika **32**, 70–80 (1941); these Rev. **2**, 236] that the D^2 test is most powerful with respect to all alternatives whose power function depends only on $\Delta^2=p^{-1}\sum_{i,j=1}^{p}\alpha^{ij}(E(x_i)-E(y_i))(E(x_j)-E(y_j))$, where $\alpha_{ij}=(\alpha^{ij})^{-1}$ is the common covariance matrix of the X's and Y's. *H. B. Mann* (Columbus, Ohio).

Welch, B. L. The generalization of 'Student's' problem when several different population variances are involved. Biometrika **34**, 28–35 (1947).

Let η be a population parameter which is estimated by an observed quantity y, normally distributed with variance $\sigma_y^2=\sum_{i=1}^{k}\lambda_i\sigma_i^2$, where the λ_i are known positive numbers and the σ_i are unknown variances. Suppose, furthermore, that the observed data provide estimates s_i^2 of the variances based upon f_i degrees of freedom, respectively. For given probability P the author seeks an $h(s_1^2, \cdots, s_k^2, P)$, such that the chance that the difference $(y-\eta)$ falls short of h is P. An exact solution is given, although some writers seem to have doubted whether the problem admits of a solution. A series in powers of $1/f_i$ is developed, available for calculating tables. It is shown how the given inequality may be adapted to provide an interval estimate for η. The relation of this study with work and views of others on this and related problems is discussed. The exact solution is expanded only through terms in $1/f_i^2$ because of the complexity of the terms. A practical approximation is also obtained. *A. A. Bennett* (Providence, R. I.).

Lord, E. The use of range in place of standard deviation in the t-test. Biometrika **34**, 41–67 (1947).

Let $x_1, \cdots, x_{mn}$ be normally and independently distributed chance variables with mean zero and variance σ^2. Without loss of generality for what follows we may take $\sigma^2=1$. Let $mn\bar{x}=\sum_{i=1}^{mn}x_i$. Let w_j be the range of the observations $x_{(j-1)n+1}, \cdots, x_{jn}$, $j=1, \cdots, m$ (this corresponds to a random division of the sample of mn into m subgroups of equal size), $d_n=Ew_j$ and $m\bar{w}(m, n)=\sum_{j=1}^{m}w_j$. Then the statistic $u(m, n)=\bar{x}d_n(mn)^{\frac{1}{2}}/\bar{w}(m, n)$ can be used essentially for the same purpose as Student's t (tests of hypotheses, estimation). The analytic form of the distribution of u is not very manageable. The present paper gives important percentage points of $q=u(m, n)/d_n$ for a number of values of m and n. The technique of the computation is described and examples of application are given.

It is proved that $\bar{x}$ and what is essentially the range of a sample are independently distributed. This, however, is a special case of a theorem of Daly [Ann. Math. Statistics **17**, 71–74 (1946); these Rev. **7**, 464], who also gave a number of the percentage points given here.

J. Wolfowitz (New York, N. Y.).

Geary, R. C. The frequency distribution of $\sqrt{b_1}$ for samples of all sizes drawn at random from a normal population. Biometrika **34**, 68–97 (1947).

Let $x_1, \cdots, x_n$ be a sample of n independent observations from a normal distribution and define

$$t_n=n^{\frac{1}{2}}\left[\sum_i(x_i-\bar{x})^3\right]\left[\sum_i(x_i-\bar{x})^2\right]^{-\frac{3}{2}},$$

where as usual $n\bar{x}=\sum_i x_i$. This paper discusses the distribution of t_n. Let $f_n(t)$ be the density of t_n. The author obtains a recursion formula for f_n in terms of f_{n-1}, and approximate forms for f_n at various parts of the range of t_n. The result for f_3 is known and simple [Fisher, Proc. Roy. Soc. London. Ser. A. 130, 16–28 (1930)]. The author discusses in considerable detail the cases $n=4, \cdots, 8$, and then derives asymptotic formulae for quantiles which he checks for deviation from the results obtained for $n=8$. The methods used do not lend themselves to succinct description.

J. Wolfowitz (New York, N. Y.).

Geary, R. C., and Worlledge, J. P. G. On the computation of universal moments of tests of statistical normality derived from samples drawn at random from a normal universe. Application to the calculation of the seventh moment of b_2. Biometrika 34, 98–110 (1947).

Elfving, G. The asymptotical distribution of range in samples from a normal population. Biometrika 34, 111–119 (1947).

The range w of a sample of n from the normal population with cumulative distribution function

$$\Phi(x)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x}\exp\left(-\tfrac{1}{2}u^2\right)du$$

is transformed to $x^*=2n\Phi(-\frac{1}{2}w)$ and it is proved that as $n\rightarrow\infty$ the cumulative distribution function of x^* has the limit $1+\frac{1}{2}\pi x H_1^{(1)}(x)$, where $H_1^{(1)}$ is a Hankel function.

H. Scheffé (Berkeley, Calif.).

Plackett, R. L. Limits of the ratio of mean range to standard deviation. Biometrika 34, 120–122 (1947).

If d_n is the ratio of the expected range in samples of n to the population standard deviation, an upper limit for d_n independent of the form of the distribution is

$$n\left[2\frac{(2n-2)!-(n-1)!^2}{(2n-1)!}\right]^{\frac{1}{2}},$$

which is approximately $n^{\frac{1}{2}}$ for large n. Distributions for which d_n attains this value are exhibited; for $n=2, 3$ the distributions are rectangular.

A. M. Mood.

Pearson, E. S. The choice of statistical tests illustrated on the interpretation of data classed in a 2×2 table. Biometrika 34, 139–167 (1947).

Three different statistical situations in which the data may be presented in a fourfold table are used to illustrate certain aspects of the Neyman-Pearson theory of testing hypotheses. The three different situations might be characterized by the appropriateness of considering fixed (I) the row totals and the column totals, (II) the row totals, (III) the grand total. The paper contains discussions covering much ground, from the role of probability calculations in making practical decisions to the adequacy of the normal approximation to the hypergeometric distribution. Among the conclusions are the following. The "correction for continuity" should be used in applying the normal approximation to problem (I) but not problem (II). The customary common treatment of all three problems by the chi-square method is acceptable for large samples.

H. Scheffé.

Barnard, G. A. 2×2 tables. A note on E. S. Pearson's paper. Biometrika 34, 168–169 (1947).

Cf. the preceding review.

Barnard, G. A. Significance tests for 2×2 tables. Biometrika 34, 123–138 (1947).

Statisticians who have puzzled over the propriety of assuming fixed marginal totals when calculating probabilities for fourfold tables will find illuminating the discussion in part I of this paper of three different probability models, each appropriate for application to a certain kind of data. In part II there is proposed a new test called the "CSM test" for the hypothesis that two binomial populations have equal parameters. The justification of the CSM procedure is largely intuitive, all probability calculations involved being made for the case when the hypothesis is true. To apply the new test with small sample sizes special tables are necessary, one double entry table for each pair of sample sizes. A sample of three such tables is offered and in one of them a comparison is made with R. A. Fisher's "exact test" based on fixed marginals.

H. Scheffé (Berkeley, Calif.).

Barnard, G. A. The meaning of a significance level. Biometrika 34, 179–182 (1947).

Barnard, G. A. Sequential tests in industrial statistics. Suppl. J. Roy. Statist. Soc. 8, 1–21 (1946).

The author outlines briefly some of the work he and his associates have done on sequential sampling and the relation of that work to the general theory as developed in America. The analogy of sequential sampling schemes to the linear diffusion problem has led the author to consider general linear sequential tests. It is pointed out that for certain types of distributions the sequential probability ratio test becomes a linear test. The reviewer would like to remark in this connection that the probability ratio test becomes a linear test in the general case if the original variable x is replaced by $z=\log\{f_1(x)/f_0(x)\}$, where $f_i(x)$ is the elementary probability law of x under the hypothesis H_i ($i=0, 1$). Two sequential sampling schemes for double dichotomies are discussed, both different from the procedure that results from the application of the sequential probability ratio test. The last part of the paper is devoted to general inspection problems. The author distinguishes between acceptance sampling and rectifying inspection. The process curve, the outgoing quality curve, the average outgoing quality and related notions are discussed. The paper is followed by discussion by Womersley, Bartlett and others.

A. Wald (New York, N. Y.).

Burman, J. P. Sequential sampling formulae for a binomial population. Suppl. J. Roy. Statist. Soc. 8, 98–103 (1946).

The author considers the sequential probability ratio sampling plan for testing the mean of a binomial population. The sampling plan may be described as follows. Start with a score H_2. For each nondefect sampled, add 1. For each defect, subtract b. Accept the batch if the score reaches $2H=H_1+H_2$ and reject the batch if the score falls to zero or below. The author assumes that b is an integer and derives the exact operating characteristic of the sampling plan, as well as the moment generating function of the sample size. The first two moments of the sample size are obtained by differentiating the moment generating function. Limiting formulas for the operating characteristic and for the expected sample size are given when the mean p of the binomial population is small.

The binomial test when b is an integer is contained as a special case in a general class of sequential tests for which the reviewer has given a general method for obtaining the

exact operating characteristic and the exact moment generating function of the sample size [Ann. Math. Statistics 15, 283–296 (1944); these Rev. 6, 88]. An explicit formula for the exact operating characteristic of the binomial test when b is an integer was derived also by M. A. Girshick, independently of the author and apparently about the same time [Ann. Math. Statistics 17, 282–298 (1946), in particular, pp. 288–291; these Rev. 8, 163]. The author uses a method different from that of Girshick and treats also the limiting case when p is small. *A. Wald.*

Stockman, C. M., and Armitage, P. Some properties of closed sequential schemes. Suppl. J. Roy. Statist. Soc. 8, 104–112 (1946).

The authors consider the sequential probability ratio test for the binomial case modified by the condition that if no decision has been reached by the time a sample of a certain size, say n_0 observations, has been taken, inspection is terminated at the n_0th observation with the acceptance or rejection of the batch depending on the value of the score at that stage. A scheme which has a finite upper limit for the sample size is called a closed sequential scheme. A method is given for deriving the operating characteristic and average sample size of a closed sequential probability ratio test. The limiting case is considered when the constant b approaches ∞ and H_1/b and H_2/b remain constant [for a definition of H_1, H_2 and b see the preceding review]. The authors use matrix methods similar to those given by W. Bartky [Ann. Math. Statistics 14, 363–377 (1943); these Rev. 5, 209] for deriving the operating characteristic and average sample size of the open scheme defined by H_1, H_2 and b. *A. Wald* (New York, N. Y.).

Linder, Arthur. Sur la manière d'organiser les expériences afin d'obtenir un rendement maximum. Arch. Sci. Phys. Nat., Geneva (5) 28, 181–191 (1946).

Lecture to the Société de Physique et d'Histoire Naturelle de Genève.

Radhakrishna Rao, C. Hypercubes of strength 'd' leading to confounded designs in factorial experiments. Bull. Calcutta Math. Soc. 38, 67–78 (1946).

Consider m factors each at s levels such that each treatment combination corresponds to a point $(i_1, \cdots, i_m)$, $0 \leq i_\alpha \leq s-1$. A hypercube (m, s, t, d) of strength d is an array of s^t points with m coordinates each such that every combination $i_{k_1}, \cdots, i_{k_d}$ occurs s^{t-d} times in it. An (m, s, t, d) can be obtained from the $PG(t, s)$, if it is possible to find in the $PG(t-1, s)$ a set of $(t-2)$-flats, m in number, such that no d of them intersect in a $(t-d)$-flat. The maximum m for given s, t, d is not known. The author gives a method for constructing hypercubes. From the hypercubes constructed by his method it is easy to obtain designs in which only interactions of order d or higher are confounded. Obviously for $d=2$ the maximum m is $(s^t-1)/s-1$ if s is a prime power. Some asymmetrical factorial designs may likewise be obtained from hypercubes of strength d.
H. B. Mann (Columbus, Ohio).

Del Chiaro, A. Sui tassi centrali di mortalità. Giorn. Ist. Ital. Attuari 12, 208–220 (1941).

On the central death rate and its relations to the probability of death. *P. Johansen* (Copenhagen).

Ottaviani, G. Sulle formule di contribuzione e sulla partecipazione degli assicurati agli utili. Giorn. Ist. Ital. Attuari 12, 191–207 (1941).

On the principle of contribution in surplus distribution.
P. Johansen (Copenhagen).

Mathematical Economics

May, Kenneth. The aggregation problem for a one-industry model. Econometrica 14, 285–298 (1946).

A one-industry model with one degree of freedom in N, w/P, U, π/P, where N is total employment, P the price of net output, U the net output, w the money wage per unit of employment and π the total money profits, is defined by the relations $\pi = PU - wN$ and $U = \psi(N)$ and the equilibrium condition $\psi'(N) = w/P$. A solution of the aggregation problem for this model is given for restricted general equilibrium systems based on those due to Dresch [Bull. Amer. Math. Soc. 44, 134–141 (1938)]. Free competition is assumed.
M. P. Stoltz (Providence, R. I.).

Schelling, Thomas C. Raise profits by raising wages? Econometrica 14, 227–234 (1946).

Let national money income Y = wages W + profit P. Let $C_w = C_w(W)$ and $C_p = C_p(P)$ be the consumption of wage-earners and profit receivers, respectively. Denoting by C_w' and C_p' the corresponding derivatives ("marginal propensities to consume"), we have ordinarily $C_w' > C_p'$, $C_p' < 1$; and since $Y = C_w + C_p$ + private investment + public expenditure, we have, if the latter two terms are constant,

$$dY/dW = (C_w' - C_p')/(1 - C_p') > 0.$$

Hence rising wages increase money income. But they increase profits only under the unlikely condition $C_w' > 1$, since only then $dP/dW = (C_w' - 1)/(1 - C_p') > 0$. The condition $C_w' > 1$ is replaced by milder but still unlikely ones if private investment or public expenditure are functions of income, or if a third social group shares in the income.
J. Marschak (Chicago, Ill.).

Ville, Jean. Sur les conditions d'existence d'une ophélimité totale et d'un indice du niveau des prix. Ann. Univ. Lyon. Sect. A. (3) 9, 32–39 (1946).

The author starts by imposing several restrictions on the demand functions for an individual: they must be functions of prices and income; if prices and income vary and the individual does not buy the same quantities as before although he could afford to, he prefers the new position; if prices and income vary in a closed cycle, he does not prefer every new position to its predecessor. From these hypotheses, it is shown that there exists a total utility function from which the demand functions could be deduced in the usual manner. [This theorem has been established by H. Wold, Skand. Aktuarietidskr. 27, 69–120 (1944), in particular, pp. 78–88; these Rev. 6, 238, though the hypotheses are not stated as clearly there.] The author then expresses the results of utility-maximization in various forms, stressing particularly a function of prices and income obtained by substituting into the utility function the optimum quantities corresponding to each price-income situation. This function is shown to enjoy properties symmetrical to the utility function. [This result was obtained earlier by L. M. Court, Econometrica 9, 135–137, 284–293 (1941).]

Defining the Divisia index of the standard of living as $\exp \int_0^t (\sum p_i dq_i / \sum p_i q_i)$, where p_i (q_i) is the price (quantity consumed) of commodity i, and the Divisia index of cost

of living as the index of income divided by the index of the standard of living, it is shown that the index is legitimate when and only when the preference scale can be defined by a utility function which is a homogeneous function of quantities consumed. *K. J. Arrow* (Chicago, Ill.).

Palomba, Giuseppe. Elementi matematici per l'economia corporativa. (Sul problema della produzione). Rend. Accad. Sci. Fis. Mat. Napoli (4) 12, 148–156 (1942).

***Amoroso, Luigi. Riflessioni sulla dinamica dei prezzi.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 677–682. Edizioni Cremonense, Rome, 1942.

Tintner, Gerhard. Some applications of multivariate analysis to economic data. J. Amer. Statist. Assoc. 41, 472–500 (1946).

Expository article with numerous applications, most of which are new. *H. Wold* (Uppsala).

TOPOLOGY

Tutte, W. T. On Hamiltonian circuits. J. London Math. Soc. 21, 98–101 (1946).

The author presents a regular map on a sphere, of 25 regions, for which Tait's conjecture on the existence of a simple circuit through all the vertices does not hold. A theorem on such circuits, due to C. A. B. Smith, is proved. *P. Franklin* (Cambridge, Mass.).

Jones, F. Burton. A characterization of a semi-locally-connected plane continuum. Bull. Amer. Math. Soc. 53, 170–175 (1947).

The author characterizes semi-locally-connected continua on S^2 as those continua whose complements are not folded. If D is an open set, then D is said to be folded provided that there exist in D mutually exclusive sets $X_1, X_2, X_3, \cdots$ such that (1) for each i ($i=1, 2, 3, \cdots$), X_i is the sum of two arc-segments T_{i1} and T_{i2} having their end points in the boundary of D, (2) for each i, T_{i1} crosses T_{i2}, (3) the sequence $T_{11}, T_{21}, T_{31}, \cdots$ converges to a subset T of $S-D$, and (4) the sequence of end points of $T_{12}, T_{22}, T_{32}, \cdots$ converges to a point of $S-T$. This result is analogous to Schoenflies' characterization of a continuous curve in R^2. A related theorem concerns cyclic continua. *A. D. Wallace* (Philadelphia, Pa.).

van Heemert, A. The existence of 1- and 2-dimensional subspaces of a compact metric space. Nederl. Akad. Wetensch., Proc. 49, 905–910 = Indagationes Math. 8, 564–569 (1946).

It is well known that a space of finite dimension has subsets of every smaller dimension. It has been shown by Hurewicz, however [Fund. Math. 19, 8–9 (1932)], that on the hypothesis of the continuum there exist infinite dimensional metric spaces (subsets of Hilbert space) whose only finite dimensional subsets are of dimension zero, in fact are countable sets; Hurewicz's example is not compact. The author demonstrates that for a compact metric space of infinite dimension there must exist subsets of dimension 1 and 2. The proof employs Freudenthal's R_n-adic developments, in particular, the fact that a compact metric space R is the limit space of an R_n-adic sequence of polytopes P_n, $P_1 \leftarrow P_2 \leftarrow \cdots (f_n^{n+1} P_{n+1} = P_n)$, where each f_n^{n+1} is normal and irreducible. It is proved that if dim $R=2$ there can be associated with this sequence an R_n-adic sequence, also normal and irreducible, ${}^dP_1 \leftarrow {}^dP_2' \leftarrow {}^dP_3' \leftarrow \cdots (f_n^{n+1}\, {}^dP'_{n+1} = {}^dP_n'$, where $d=1$ or 2, ${}^dP$ denotes the d-dimensional skeleton of P, $P_2' = (f_1^2)^{-1}\, {}^dP_1$, $P_3' = (f_2^3)^{-1}\, {}^dP_2'$, $\cdots$ and each element of the sequence is not empty. The limit space of this sequence is then a compact subset of R having dimension d. *H. Wallman* (Cambridge, Mass.).

Scorza Dragoni, G. Ancora sugli archi di traslazione di un autoomeomorfismo piano privo di punti uniti e conservante il senso delle rotazioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 918–922 (1946).

Let t be a fixedpoint-free, sense-preserving topological automorphism of the (x, y)-plane. Let

$$\sigma = \cdots + t^{-1}(\lambda) + \lambda + t(\lambda) + \cdots$$

be a simple trajectory of t, λ being a "translation arc." Let Σ be one of the two regions adjacent to σ. Then either there exists in Σ a simple arc l with initial point on σ and open in the other direction (containing, in fact, a ray parallel to one of the axes) with the property that it fails to intersect its image under t, or there is in Σ a linear segment parallel to one of the axes, which is a translation arc of t and which is contained in a "pseudo-translation arc" (limit of translation arcs) lying in Σ, except for initial point on σ. *P. A. Smith* (New York, N. Y.).

Whitney, Hassler. Algebraic topology and integration theory. Proc. Nat. Acad. Sci. U. S. A. 33, 1–6 (1947).

The note gives an outline of a theory of integration in terms of the cohomology theory in algebraic topology. The treatment is carried out largely without using coordinate systems. A so-called Lipschitz mapping plays an important rôle; this is a mapping of one metric space into another satisfying a Lipschitz condition: dist $[f(p), f(q)] \leqq N$ dist (p, q) for some N. The most general space to which the results apply is a Lipschitz space R, that is, a metric space such that (1) it is imbeddable into some Euclidean space by a mapping f which, together with its inverse, is Lipschitz; (2) $f(R)$ is locally Lipschitz connected in each dimension. Lipschitz chains and cochains are defined, also the coboundary of a Lipschitz cochain. The resulting Lipschitz cohomology groups are proved to be dimensionwise isomorphic to the algebraic cohomology groups (with real coefficients). As generalization of the integrands of multiple integrals (which are alternating differential forms or alternating tensors) Lipschitz tensor cochains are defined. The tensor cohomology groups are again proved to be isomorphic to the algebraic ones. The direct sums of the respective cohomology groups, algebraic, Lipschitz, and Lipschitz tensorial, are turned into rings, by the introduction of a multiplication operation. It is proved that the cohomology rings thus resulting are mutually isomorphic. *S. Chern* (Shanghai).

Whitney, Hassler. Geometric methods in cohomology theory. Proc. Nat. Acad. Sci. U. S. A. 33, 7–9 (1947).

Geometric interpretations are given to cochains and cocycles. The topological spaces in which the interpretations hold are polyhedra but some of the results hold also for

absolute neighborhood retracts. A subset N of the space R is called r-thin if any singular r-cell in R may be pulled away from N by an arbitrarily small deformation. Given a discrete coefficient group H, a geometric r-H-cochain $X=X^r$ in R is defined in terms of a nucleus $\bar{X}$ and a nuclear boundary $\bar{X}'\subset\bar{X}$, both of which are closed sets and are $(r-1)$-thin and r-thin, respectively, such that $X^r\cdot A^r\epsilon H$ is defined for any singular r-chain A^r, with integer coefficients, which satisfies the conditions $A^r\cap\bar{X}'=0$, $\partial A^r\cap\bar{X}=0$. Furthermore, the following properties are supposed: (a) $X^r\cdot A^r=0$ if A^r does not intersect $\bar{X}$; (b) $X^r\cdot\partial A^{r+1}=0$ if A^{r+1} does not intersect $\bar{X}'$. The coboundary is defined and the geometric cohomology groups so obtained are proved to be dimensionwise isomorphic to the algebraic cohomology groups. Various properties of geometric cochains are given. In particular, the product of geometric cochains is defined and its usual properties are established. *S. Chern.*

Whitney, Hassler. Complexes of manifolds. Proc. Nat. Acad. Sci. U. S. A. **33**, 10–11 (1947).

The complexes of manifolds, or complifolds, are intended to be the boundaries of bounded manifolds. Such a boundary K can be a quite general point set and discussions are given of the conditions to bring it back to reasonableness. A guiding principle is that K is imbeddable in a Euclidean space, each cell being imbedded smoothly. By consideration of the tangent spaces of K conditions are obtained which are either necessary or sufficient for K to be smoothly imbeddable. In particular, by a notion called cellwise homogeneity it is proved that any cellwise homogeneous complifold of dimension n may be smoothly imbedded in a Euclidean space of dimension $2n+1$ and smoothly immersed in a Euclidean space of dimension $2n$. *S. Chern.*

Eilenberg, Samuel, and MacLane, Saunders. Determination of the second homology and cohomology groups of a space by means of homotopy invariants. Proc. Nat. Acad. Sci. U. S. A. **32**, 277–280 (1946).

Let π_1, π_2 be groups with π_2 Abelian and π_1 operating on π_2. Let $\mathbf{k}^3(w_1, w_2, w_3)$ be a cocycle of π_1 with coefficients in π_2 [see the authors' paper in Ann. of Math. (2) **48**, 51–78 (1947); these Rev. **8**, 367] and let G be a topological Abelian group. The authors construct from these elements an additive group $H^2(\pi_1, \pi_2, \mathbf{k}^3, G)$. It is stated that, if π_1, π_2 are the first two homotopy groups of an arcwise connected space X with π_1 operating on π_2 in the usual manner, and if $\mathbf{k}^3$ is a certain new homotopy invariant which the authors obtain by a construction which involves mappings of the faces of a 3-simplex into singular cells of X defined with the aid of arbitrary elements w_1, w_2, w_3 of π_1, then the group H^2 is isomorphic to the 2-dimensional cohomology group of X over G. *P. A. Smith* (New York, N. Y.).

GEOMETRY

Banning, J. On the order in an Euclidean plane. Nieuw Arch. Wiskunde (2) **22**, 115–122 (1946).

In his "Grundlagen der Geometrie" [7th ed., Leipzig, 1930] Hilbert proved (1) to two points A, C there exists a point B of the line AC which is between A and C, (2) of any three points on a line, one is between the other two, (3) any four points of a line may be labelled A, B, C, D so that B is between A, C and also between A, D, while C is between A, D and B, D, and (4) the generalization of (3) for n points of a line. The author proves these four theorems in a plane geometry based upon (1) Hilbert's three plane axioms of connection and (2) a weakened set of order axioms obtained by substituting for Hilbert's axiom II2 of external convexity the assumption that each line contains at least three points and there exist three points A, B, P with P between A and B. *L. M. Blumenthal* (Columbia, Mo.).

Walker, Richard. Die Hilbertschen Axiome der Geometrie und ihre gegenseitige Unabhängigkeit. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **95** (1943), 151–170 (1944).

In his "Grundlagen der Geometrie" [7th ed., Leipzig, 1930] Hilbert proved the independence of congruence axiom III5 [Euclid I, 4], the parallel axiom and each of the continuity axioms V1, V2. The aim of the present paper is to investigate the question of independence for each axiom of the system. To this end the author finds it convenient to modify Hilbert's system slightly and formulate it in terms of nine axioms of connection, five axioms of order, seven congruence axioms (the Hilbert system has eight, four, and five axioms, respectively, in these three groups), the parallel axiom and one continuity axiom [Dedekind] which replaces the Archimedean and linear completeness assumptions of Hilbert. The paper furnishes independence proofs for all but two of the axioms. The exceptions (whose dependence or independence are not established) are, in the author's system, congruence axiom III3 (if $AB\equiv A'B'$, $BC\equiv B'C'$, with B and B' between A, C and A', C', respectively, then $AC\equiv A'C'$) and congruence axiom III6, which is equivalent to the assumption that an angle is congruent with itself. *L. M. Blumenthal* (Columbia, Mo.).

Bouwkamp, C. J. On the dissection of rectangles into squares. I. Nederl. Akad. Wetensch., Proc. **49**, 1176–1188 = Indagationes Math. **8**, 724–736 (1946).

Bouwkamp, C. J. On the dissection of rectangles into squares. II, III. Nederl. Akad. Wetensch., Proc. **50**, 58–71, 72–78 = Indagationes Math. **9**, 43–56, 57–63 (1947).

Brooks, Smith, Stone and Tutte [Duke Math. J. **7**, 312–340 (1940); these Rev. **2**, 153] discussed the partition of a rectangle into a finite number n of disjoint squares; n was called the order of the squaring. Their method consisted in associating an electric network with each squaring. This paper contains some comments and amplifications. The following may be mentioned. (1) A short notation for squarings is introduced. (2) The associated network is constructed in a very simple (though not essentially new) fashion. (3) The numbers of the different types of squarings of order not exceeding 13 are given. (4) The 354 "simple" squarings of order not exceeding 13 are enumerated [a squaring is "simple" if no proper subset of the set of its squares, each in its original position, forms a rectangle]. (5) In the paper quoted a method was outlined for constructing a "simple" partition of a square into mutually different squares. This method is criticized. *P. Scherk.*

✶Danielsson, Gösta. Les sphères circonscrite et inscrite à un tétraèdre. C. R. Dixième Congrès Math. Scandinaves 1946, pp. 352–355. Jul. Gjellerups Forlag, Copenhagen, 1947.

A formula of Euler states that $d^2=R(R-2r)$, where R, r are the radii of the circumscribed and inscribed circles of the triangle and d the distance of their centers. Durrande

stated that for the tetrahedron $d^2=(R+r)(R-3r)$. The author shows that this is not correct and that no algebraic relation exists between d, R and r alone. He also shows among other things that $R\geqq 3r$ and conjectures that $d^2\leqq(R+r)(R-3r)$. *P. Erdös* (Syracuse, N. Y.).

Hohenberg, Fritz. Das Apollonische Problem im R_n. Deutsche Math. 7, 78–81 (1942).

The problem of finding the hyperspheres $\mathfrak{K}$ (centre $M=(\xi_1,\cdots,\xi_n)$, radius ρ) which touch $n+1$ given hyperspheres $\mathfrak{K}_i$ ($i=1,\cdots,n+1$; centre $M_i=(m_{i1},\cdots,m_{in})$, radius r_i) was solved by Schoute [Akad. Wetensch. Amsterdam, Proc. 7, 562–572 (1905); Mehrdimensionale Geometrie, v. 2, Leipzig, 1905, p. 281], who determined the centres M of the required hyperspheres. The author solves the problem (for the general case where the M_i are not in a hyperplane) by first finding their radii ρ in the following way. Put $\epsilon_i=\pm 1$ according to whether $\mathfrak{K}$, $\mathfrak{K}_i$ touch externally or internally. For any combination $(\epsilon_1,\cdots,\epsilon_{n+1})$ the conditions of the problem are

$$\sum_{\lambda=1}^{n}(\xi_\lambda-m_{i\lambda})^2-(\rho+\epsilon_i r_i)^2=0,\quad i=1,2,\cdots,n+1.$$

By subtraction n linear equations are formed and elimination of $\xi_1,\cdots,\xi_n$ yields a quadratic equation for ρ which may be written invariantly as

$$(\bar{I}^2-2I^2)\rho^2-2(n+1)IJ\rho+I^2t^2=0, \tag{1}$$

where t^2 is the power with respect to the $\mathfrak{K}_i$ of their radical centre O, I the n-dimensional volume of the simplex $M_1M_2\cdots M_{n+1}$, $\bar{I}$ the n-dimensional volume of the simplex $K_1K_2\cdots K_{n+1}$, where K_i is the Laguerre image of $\mathfrak{K}_i$ in R_{n+1}, i.e., the point $(m_{i1},\cdots,m_{in},\epsilon_i r_i)$, and J the $(n+1)$-dimensional volume of $O'K_1\cdots K_{n+1}$, where O' is the Laguerre image of the sphere with centre O and radius 0. There are 2^n essentially different equations (1) (as the system $-\epsilon_i$ leads to the radius $-\rho$ which is geometrically equivalent to ρ) and, hence, 2^{n+1} values of ρ; for each ρ the ξ_ν are determined by linear equations. *F. A. Behrend.*

✶**Di Marco, Luigi. Sulle risoluzioni grafiche dei problemi di Snellius-Pothenot e di Hansen.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 951–954. Edizioni Cremonense, Rome, 1942.

✶**Natucci, A. Vari tipi di dimostrazione del teorema fondamentale della geometria proiettiva.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 977–982. Edizioni Cremonense, Rome, 1942.

Manara, Carlo Felice. Vedute sulla geometria del triangolo. Period. Mat. (4) 22, 145–157 (1942).

Expository article emphasizing projective methods.

Bogdan, C. P. Sur l'enveloppe des familles planes de coniques. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 1, 289–291 (1946).

Bogdan, C. P. Sur les familles de coniques de l'espace ordinaire. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 1, 292–294 (1946).

✶**Kàrteszi, Francesco. Su un sistema speciale delle coniche osculatrici a due coniche date.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 271–276. Edizioni Cremonense, Rome, 1942.

✶**Kàrteszi, Francesco. Sulla parabola di Neil.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 994–998. Edizioni Cremonense, Rome, 1942.

Pernet, Roger. Un principe de traduction de propriétés géométriques de la droite et du cercle, en propriétés des séries de Villarceau d'une congruence paratactique. C. R. Acad. Sci. Paris 221, 601–603 (1945). [MF 15145]

Rao, C. V. Hanumanta. On the generation of sets of four tetrahedra of which any two are mutually inscribed. Proc. Cambridge Philos. Soc. 42, 217–226 (1946).

Baker, H. F. Note to the preceding paper by C. V. H. Rao. Proc. Cambridge Philos. Soc. 42, 226–229 (1946).

These papers are concerned with Kummer's configuration (16_6). It is well known that the 16 points and the 16 planes of this configuration can be arranged into 4 tetrahedra of which every two are mutually inscribed. It is shown that these four tetrahedra may be derived from a pair of suitably chosen tetrahedra (X,Y,Z,T) and (P,Q,R,S). Taking the harmonic inverse of (P,Q,R,S) with respect to T and the plane XYZ one tetrahedron is obtained. Similarly three more tetrahedra are constructed by inversion on the remaining three vertices of (X,Y,Z,T). The four tetrahedra so obtained are such that every two are mutually inscribed. The pair of tetrahedra (X,Y,Z,T) and (P,Q,R,S) may be chosen in 24 different ways. In the first paper a geometric proof is given for this generation of the four mutually inscribed tetrahedra; in the second paper the same statement is proved analytically. *E. Lukacs.*

Sydler, J.-P. Des hyperquadriques et droites associées de l'espace à n dimensions. Comment. Math. Helv. 19, 161–214 (1947).

L'auteur étend aux espaces linéaires d'un espace projectif E^n à n dimensions, dans lequel on introduit une métrique cayleyenne au moyen d'une hyperquadrique non dégénérée prise pour absolu, certaines propriétés des droites associées de Schläfli [J. Reine Angew. Math. 65, 189–197 (1866)]. Deux points sont inverses par rapport à un simplexe, s'ils sont les foyers d'une hyperquadrique de révolution tangente aux faces de ce simplexe. Cette correspondance transforme $n+1$ droites associées passant par les sommets du simplexe en $n+1$ droites associées.

On dit que $n-k+2$ sous-espaces E^k de E^n sont associés si tout E^{n-k-1} incident à $n-k+1$ d'entre eux est encore incident au dernier. Dans la variété Grassmannienne représentative des E^k, les espaces associés sont représentés par des points linéairement dépendants. Pour que $n-k+2$ sous-espaces E^k soient associés il faut et suffit que leurs sections par les E^{n-k+1} d'un simplexe arbitraire soient formées de droites associées. Signalons encore ce résultat: l'espace à 4 dimensions est le seul espace dans lequel un groupe de droites détermine univoquement celles qui doivent leur être associées: on a alors la configuration classique de Segre [Rend. Circ. Mat. Palermo 2, 45–52 (1888)]. *L. Gauthier.*

✶**Fabricius-Bjerre, Fr. Sur les courbes gauches du quatrième ordre.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 65–69. Jul. Gjellerups Forlag, Copenhagen, 1947.

The closed differentiable curves C_4 of real order 4 in projective 3-space have been classified into ten types according to the numbers of the various kinds of singularities they possess [Scherk, Ann. of Math. (2) 46, 68–82 (1945); these Rev. 6, 183]. The author proves the following theorem.

Given a closed differentiable curve K_4 of real order 4 in projective 4-space, then to each of the ten types of C_4's there are points in 4-space so that the projections of the K_4 from them are C_4's of that type. The proof is based on a discussion of the K_4 that seems of independent interest. Since the K^4 can be chosen nonalgebraic and even nonanalytic, the author's theorem implies the existence of nonanalytic C_4's of each type. This appears remarkable in view of Courtand's result that the C_4 without singularities (type IX of the classification) necessarily lies on a hyperboloid [Sur les courbes gauches du troisième et du quatrième ordre en géométrie finie, Actualités Sci. Ind., no. 868, Hermann, Paris, 1940; these Rev. 7, 526]. [The theorem should not be understood to mean that any given C_4 is the projection of a suitable K_4. Indeed, if the classification of the C_4's is refined so as to distinguish between curves of different class and rank numbers, subtypes, e.g. of type I, are obtained that cannot be such projections.] *P. Scherk.*

Smart, W. M. On a problem in navigation. Monthly Not. Roy. Astr. Soc. 106, 124–127 (1946).

When the distance between the points F (latitude ϕ_1, longitude λ_1) and $T(\phi_2, \lambda_2)$ on a rhumb line is considerable, the departure between these points is related to the difference in longitude by the formula Departure $=(\lambda_2-\lambda_1)\cos l$; l, called the "middle latitude," is given in terms of the mean latitude $\frac{1}{2}(\phi_1+\phi_2)$ by $l=\frac{1}{2}(\phi_1+\phi_2)+c$, in which c is, in general, a comparatively small calculable angle depending on the co-ordinates of F and T. When the Earth is regarded as a sphere, the method of deriving l, or c, is well known. The question has recently been raised as to the general formula for l, or c, when the spheroidal nature of the Earth is taken into account; this note is intended to supply an answer. *Author's summary.*

Algebraic Geometry

★Baker, H. F. A Locus with 25920 Linear Self-Transformations. Cambridge Tracts in Mathematics and Mathematical Physics, no. 39. Cambridge, at the University Press; New York, The Macmillan Company, 1946. xi+107 pp. $2.00.

This monograph gives a connected account, with emphasis on the geometrical point of view, of the representation of the simple group of order 25920 as a collineation group in four dimensions. The geometry centres round a quartic primal and a set of forty-five points, nodes of the primal, which are invariant under the group [see Burkhardt, Math. Ann. 38, 161–224 (1891)] and Coble [Amer. J. Math. 28, 333–366 (1906); Trans. Amer. Math. Soc. 18, 331–372 (1917)]. The early part of the monograph contains a detailed description of the figure of 45 nodes and various associated linear spaces containing subsets of them; noteworthy among these are forty-five "Jordan primes" each containing twelve nodes forming the vertices of three desmic tetrahedra, forty "Jacobian planes" each containing nine nodes which are the inflexions of a pencil of plane cubics, and forty "Steiner solids" each containing four Jacobian planes and eighteen of the nodes. The author obtains various simple forms for the equations of the quartic primal. The later part of the work is concerned with the group of collineations which leave the figure invariant, and it is shown that this group is generated by a set of harmonic inversions each of which has a node and a Jordan prime as invariant spaces. Applications are made to some of the more notable subgroups of the complete group of order 25920.

J. A. Todd (Cambridge, England).

Brown, L. M. The configuration determined by five generators of a quadric threefold. Proc. Edinburgh Math. Soc. (2) 7, 183–195 (1946).

The condition for five lines of [4] space to lie on a quadric primal is worked out geometrically and algebraically. Its necessity was discovered by H. W. Richmond [Proc. Cambridge Philos. Soc. 10, 210–212 (1900)]. B. Segre proved its sufficiency [Amer. Math. Monthly 52, 119–131 (1945); these Rev. 6, 215]. New proofs are given here. Richmond's criterion is that, if 1, 2, 3, 4, 5 denote the points where the lines α, β, γ, δ, ϵ, respectively, meet the primes through ϵ, β; α, γ; etc., respectively, then these five points must lie in a prime. The lines $\alpha, \beta, \gamma, \delta, \epsilon$ then lie on a quadric. Now one such prime corresponds to one cyclic order 12345; all possible cyclic orders lead to twelve primes, which group into six pairs such as (12345) and (13524). (Compare an ordinary, and a star, pentagon.) The primes of one pair meet in a plane; thus we have six planes, which meet by pairs in lines. Properties of the configuration so set up are developed, including a duality which arises by taking, instead of the original five lines, certain other five which meet sets of three original lines. These give six further pairs of primes, but the same six planes. The duality is occasioned by reciprocation in the quadric primal.

H. W. Turnbull (St. Andrews).

Cartan, Élie. Quelques remarques sur les 28 bitangentes d'une quartique plane et les 27 droites d'une surface cubique. Bull. Sci. Math. (2) 70, 42–45 (1946).

This note connects a classical result in the geometry of a cubic surface and a plane quartic curve with the theory of groups. Project the 27 lines lying upon a general cubic surface from a point O of the surface onto a plane π which does not contain O. Draw the tangent cone from O and also the tangent plane at O to the cubic surface. This cone cuts π in a general quartic curve which has for its double tangents the 27 lines that are the projections of the former lines, together with the line common to π and the tangent plane, 28 lines in all. Conversely, starting with the quartic curve and its 28 double tangents any one of them may be singled out from the remaining 27 and a cubic surface may be obtained. This gives a geometrical representation of a certain simple Lie group of rank 7 and order 133 which has a subgroup of rank 6 and order 78, the former group being isomorphic to the Galois group corresponding to the 28 double tangents of the quartic curve, and the latter being isomorphic to that of the 27 lines on the cubic surface. Expressions are given which generate the groups and which are closely akin to the notation of Schläfli in the classical theory. *H. W. Turnbull* (St. Andrews).

Berzolari, Luigi. Sui combinanti dei sistemi di forme binarie annessi alle curve razionali. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 14, 545–601 (1943).

On sait que si l'on donne les équations paramétriques d'une courbe rationnelle C_d^n de l'ordre n dans un espace S_d à d dimensions: $x_i=f_i(\lambda)=a_{i0}\lambda^n+\cdots+a_{in}$, $i=0, 1, \cdots, d+1$, on peut former (de deux manières différentes) les "combinants élémentaires" des $d+1$ formes binaires $f_i(\lambda)$ [voir, par exemple, K. Rohn et L. Berzolari, Encyklopädie Math. Wiss., III C 9, § 55]; la considération de ces combinants

est fondamentale dans l'étude de la courbe C_d^n par les méthodes du calcul symbolique des formes binaires. Dans son mémoire l'auteur donne en premier lieu (§ 1) une quantité d'interprétations géométriques des différents combinants élémentaires qui appartiennent à la courbe C_d^n pour des valeurs quelconques de n, d. Dans le § 2 il considère le cas, où $n=d$, d'une courbe rationnelle normale. Les § 3 et 4 sont dédiés aux courbes rationnelles C_n^{n+1} de l'ordre $n+1$ dans un espace à n dimensions; surtout remarquable est ici la considération d'un certain système linéaire de quadriques lié à la courbe C_n^{n+1} et qui est la généralisation d'un faisceau de quadriques que l'auteur même a déjà trouvé dans le cas $n=3$ [loc. cit., Anm. 377]. Enfin, dans les § 5 et 6, l'auteur s'occupe des courbes rationnelles C_n^{n+2} de l'ordre $n+2$ dans un espace à n dimensions; il les obtient, en particulier, en supposant que les formes $f_i(\lambda)$ soient les dérivées de l'ordre n d'une même forme binaire du degré $2n+2$; cette méthode avait été déjà appliquée par différents auteurs dans le cas $n=2$, et par l'auteur même dans le cas $n=3$ [Atti Accad. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (4) **7**, 305–341 (1890)].

E. G. Togliatti (Gênes).

Berzolari, Luigi. Alcune osservazioni sopra un teorema di H. Grassmann relativo alla generazione proiettiva delle curve piane algebriche. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **8(77)**, 240–248 (1944).

Berzolari, Luigi. Sul gruppo ottaedrico di collineazioni piane. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **9(78)**, ~~402~~–424 (1945).
401

Longhi, Ambrogio. Contributo alla geometria sulle curve ellittiche. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **12**, 259–278 (1942).

L'auteur considère dans ce mémoire les involutions elliptiques existant sur une courbe elliptique générale C; ses recherches se rattachent à des recherches antérieures de G. Castelnuovo [Atti Accad. Torino **24**, 4–22 (1888)] et de R. Torelli [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) $\mathbf{21}_1$, 453–457 (1912)]. Il trouve avant tout des conditions nécessaires et suffisantes pour que deux involutions elliptiques appartenant à C soient birationnellement équivalentes; on a, par exemple, que deux involutions primitives (c'est-à-dire engendrées par des transformations de la 2^e espèce de C en elle même qui soient cycliques et primitives) ne peuvent être birationnellement identiques sans coincider; et deux involutions du même ordre ne peuvent, de même, être birationnellement identiques sans coincider; une involution quelconque donnée sur C est birationnellement identique à une seule involution primitive; etc. L'auteur démontre ensuite que le nombre de toutes les involutions elliptiques d'ordre n existant sur C est égal à la somme de tous les diviseurs de n (1 et n y compris). Il donne enfin une règle pour construire un ensemble de r involutions elliptiques qui soient entre elles birationnellement identiques et dont les ordres soient égaux à des nombres donnés n_i; si l'on appelle δ le plus grand diviseur commun des nombres n_i, le nombre de tous ces groupes est égal à la somme de tous les diviseurs de δ (y compris 1 et δ).

E. G. Togliatti (Gênes).

Bogdan, C. P. Sulla superficie di Veronese. Ann. Sci. Univ. Jassy. Sect. I. **26**, 335–389 (1940).

La surface de Veronese a fait l'objet d'études devenues classiques [Bertini, Introduzione alla Geometria Proiettiva degli Iperspazi, ed. 2, Messina, 1923, chap. 15 et 16]. Dans le présent mémoire l'auteur étudie spécialement la variété V_3^3 des plans tangents à la surface de Veronese F le long d'une conique, et le cône formé par les plans des coniques qui passent par un point donné, puis des générations homographiques de la surface, en vue de la détermination des surfaces de Veronese qui sont astreintes à passer par certaines courbes. Il existe une surface F unique passant par une sextique et un point donné. La solution est de même unique lorsqu'on se donne une quartique et une conique tangentes en un point, ou trois coniques passant par un point avec des tangentes coplanaires. Par une quartique et une conique ayant deux points communs distincts, ou par trois coniques deux à deux incidentes, il passe deux surfaces de Veronese.

L. Gauthier (Nancy).

Bompiani, Enrico. Una proprietà caratteristica dei coni di Veronese. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 447–453 (1943).

On sait que toute variété algébrique V d'ordre n^r et à $r+i$ dimensions, appartenant à un espace à $\binom{n+r}{r}+i-1$ dimensions, dont les sections par les espaces à $\binom{n+r}{r}-1$ dimensions soient des variétés $V_r^{n^r}$ représentant les hypersurfaces de l'ordre n d'un espace à r dimensions doit être un cône; le sommet de ce cône a la dimension $i-1$ [G. Scorza, Rend. Circ. Mat. Palermo **28**, 400–401 (1909)]. L'auteur démontre ici que l'on peut arriver au même résultat en faisant sur la variété donnée V des hypothèses bien plus générales; par exemple, il n'est pas nécessaire de supposer que V soit une variété algébrique. Plaçons-nous, par exemple, avec l'auteur, dans le cas $r=n=2$, $i=1$; il s'agit alors d'une variété V_3^4 d'un espace S_6 qui est découpée par les hyperplans de S_6 suivant des surfaces de Veronese; le théorème de G. Scorza nous dit alors que V_3^4 est un cône projetant d'un point une surface de Veronese. On parvient au même résultat en supposant seulement que V_3^4 soit le lieu de ∞^1 surfaces de Veronese se touchant toutes le long d'une conique, pourvu que les points de cette conique soient réguliers pour V_3^4; cette dernière condition est la plus importante et se rapporte à des caractères topologiques de V_3^4 que l'auteur se propose d'exposer dans un autre mémoire. Les généralisations successives aux cas: $i>1$ et $n=r=2$; $i>1$, $n>2$ et $r=2$; et au cas général ne présentent pas de difficultés.

E. G. Togliatti (Gênes).

Bompiani, E. Ricerche sugli spazi lineari di una ipersuperficie algebrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 16–18 (1946).

***Bompiani, E. Sui tipi cremonianamente distinti di fasci di Halphen con i punti base sopra una cubica razionale.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 241–246. Edizioni Cremonense, Rome, 1942.

***Campedelli, Luigi. La classificazione dei piani doppi con tutti i generi uguali all'unità.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 248–253. Edizioni Cremonense, Rome, 1942.

***Battaglini, Francesco. Nuove formule per la risoluzione di problemi numerativi su coniche.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 259–263. Edizioni Cremonense, Rome, 1942.

***Conforto, Fabio. Sulle singolarità di alcune superficie algebriche.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 264–270. Edizioni Cremonense, Rome, 1942.

***Beloch, Margherita Piazzolla. Sulle secanti multiple delle curve algebriche sghembe.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 277–280. Edizioni Cremonense, Rome, 1942.

***Amodeo, F. Sul problema della gonalità.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 283–286. Edizioni Cremonense, Rome, 1942.

***Ales, Maria. Recenti risultati sulle forme algebrico-differenziali e loro applicazioni.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 287–290. Edizioni Cremonense, Rome, 1942.

***Buzano, Piero. Proprietà proiettive delle deformazioni di specie superiore delle varietà a 3 dimensioni dedotte col metodo della "varietà figurativa" del Bompiani.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 291–297. Edizioni Cremonense, Rome, 1942.

***Franchetta, A. Sulla caratterizzazione delle curve eccezionali riducibili di prima specie.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 303–304. Edizioni Cremonense, Rome, 1942.

Franchetta, A. Sui punti doppi isolati delle superficie algebriche. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 49–57 (1946).
Franchetta, A. Sui punti doppi isolati delle superficie algebriche. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 162–168 (1946).

Amato, Giuseppe. Sulle omografie dello spazio ordinario permutabili con una polarità non specializzata. Atti Accad. Peloritana. Cl. Sci. Fis. Mat. Nat. (3) 4(46), 188–190 (1944).

Amato, Giuseppe. Sulle omografie dello spazio ordinario che mutano in sè quadriche unite non specializzate. Atti Accad. Peloritana. Cl. Sci. Fis. Mat. Nat. (3) 4(46), 191–203 (1944).

Brusotti, Luigi. Sull'ordine di connessione delle superficie algebriche reali. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 9(78), 360–366 (1945).

Chisini, Oscar. Sulla identita birazionale delle funzioni algebriche di due variabili dotate di una medesima curva di diramazione. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 8(77), 339–356 (1944).

Galafassi, V. E. Sulle curve algebriche reali delle rigate razionali a generatrici reali. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 827–831 (1946).
Galafassi, V. E. Sulle curve algebriche reali delle rigate razionali a generatrici reali. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 922–927 (1946).

Gugliada, Maria Pia. I tipi di superficie cubica generale reale dedotti per "piccola variazione" di una rigata cubica reale. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 8(77), 15–23 (1944).

Manara, Carlo Felice. Invarianti per trasformazioni puntuali regolari dei rami superlineari ordinarii delle curve algebriche piane. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 8(77), 420–426 (1944).

Manara, Carlo Felice. Sul significato geometrico di alcuni invarianti dei rami superlineari ordinarii delle curve algebriche piane. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 8(77), 515–520 (1944).

Manara, Carlo Felice. Normale proiettiva e normale puntuale dei rami superlineari delle curve piane. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 8(77), 579–586 (1944).

Manara, Carlo Felice. Al rappresentazione analitica di una funzione algebrica di due variabili nell'intorno di una singolarità ordinaria della sua curva di diramazione. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 9(78), 191–203 (1945).

Pompilj, G. Sui piani tripli birazionalmente identici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 318–322 (1946).

Pompilj, G. Sulla equivalenza numerativa degli elementi uniti di una trasformazione cremoniana tra piani sovrapposti. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 576–580 (1946).
Pompilj, G. Sulla equivalenza numerativa degli elementi uniti di una trasformazione cremoniana tra piani sovrapposti. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 719–724 (1946).

Pompilj, G. Sui piani tripli con un fascio irrazionale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 306–313 (1946).

Pompilj, G. Sulla rappresentazione algebrica dei piani multipli. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 580–582 (1946).

Salini, Ugo. Un metodo per dedurre complessivamente gli enti proiettivamente legati ad un punto di una superficie. Atti Accad. Peloritana. Cl. Sci. Fis. Mat. Nat. (3) 4(46), 132–139 (1944).

Turri, Tullio. Condizioni perchè il gruppo di moltiplicabilità di una matrice di Riemann non risulti determinato dalle forme alternate della matrice. Rend. Sem. Fac. Sci. Univ. Cagliari 15, 66–77 (1946).

Turri, Tullio. Sulle sostituzioni modulari ammesse dalle superficie iperellittiche del tipo V secondo Scorza. Rend. Sem. Fac. Sci. Univ. Cagliari 15, 78–89 (1946).

Villa, Mario. Sulle trasformazione puntuali con direzioni caratteristiche coincidenti. I. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 9(78), 321–335 (1945).

Villa, Mario. Ricerche locali sulle trasformazioni cremoniane. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 9(78), 66–78 (1945).

Differential Geometry

De Cicco, John. Geodesic perspectivities upon a sphere. Amer. Math. Monthly **54**, 142–147 (1947).

The following fundamental theorem is proved. If a surface admits a perspectivity upon a sphere from a given point such that all the geodesics of the surface correspond to the great circles of the sphere, then the surface is a sphere (or plane) homothetic to the original with respect to the point of perspectivity. *N. A. Hall.*

✶Tonolo, Angelo. Contributo alla trigonometria dei piccoli triangoli curvilinei tracciati sopra una superficie. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 541–552. Edizioni Cremonense, Rome, 1942.

Mishra, Ratan Shanker. A note on Bianchi congruence. Bull. Calcutta Math. Soc. **38**, 141–142; erratum, 207 (1946).

It is known that on the focal surfaces of a Bianchi congruence the lines of curvature correspond and so do the asymptotic lines. The author proves that the characteristic lines correspond also. [The erratum corrects the author's name, which was originally given as Rajen Shanker Mishra.] *J. E. Wilkins, Jr.*

Simonart, Fernand. Sur les congruences pseudosphériques. Ann. Soc. Sci. Bruxelles. Sér. I. **60**, 202–206 (1946).

Les congruences pseudosphériques sont définies par la double condition que les distances entre les points limites et entre les foyers d'un rayon régulier demeurent constantes. Lorsque la congruence pseudosphérique est normale, les surfaces trajectoires orthogonales aux rayons sont des surfaces W pour lesquelles la différence entre les rayons principaux de courbure r_1, r_2 est constante. Sur les deux nappes de la développée les asymptotiques se correspondent par arcs égaux. D'après Lie, cette dernière propriété est caractéristique des surfaces W telles que $r_1-r_2=$constante. L'auteur prouve que la propriété de Lie subsiste pour une congruence pseudosphérique quelconque. Toute congruence W, telle que (1) la distance focale est constante et (2) les asymptotiques se correspondent par arcs égaux sur les deux nappes focales, est une congruence pseudosphérique. *J. Haantjes* (Amsterdam).

✶Maxia, A. Geometria proiettiva differenziale dei complessi anolonomi di rette. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 303–304. Edizioni Cremonense, Rome, 1942.

Mayer, O. Sur les congruences de droites. Ann. Sci. Univ. Jassy. Sect. I. **26**, 613–625 (1940).

On considère la congruence C définie en coordonnées projectives homogènes par $x=a(u,v)+rb(u,v)$ et une famille R de surfaces transversales découpant sur les droites de C des divisions homographiques, famille définie par une équation de la forme $dr+\alpha(r)=dr+\alpha_0r^2+2\alpha_1r+\alpha_2=0$, où les α_i sont des pfaffiens $\alpha_i=\alpha_{i1}du+\alpha_{i2}dv$. Étant donné en plus un couple f de transversales distinctes $f(r)=f_0r^2+2f_1r+f_2=0$, l'auteur, utilisant des résultats obtenus dans le problème analogue concernant les surfaces réglées, forme les invariants de la famille R et du couple f relatifs aux transformations $\bar{u}=\bar{u}(u,v)$, $\bar{v}=\bar{v}(u,v)$, $\bar{r}=(mr+n)/(pr+q)$, $\bar{f}_i=\rho f_i$. Soient M_1, M_2 et M_1', M_2' les points de rencontre des rayons (u,v) et $(u+du, v+dv)$ avec les surfaces du couple f. Les surfaces R passant par M_1', M_2' recoupent M_1M_2 en N_1 et N_2. Le premier invariant introduit, $\theta_\alpha f$, est la partie principale de $-4\log(M_1N_1M_2N_2)$. Diverses applications sont traitées.

On considère ensuite deux familles de transversales (α) et (β): $dr+\alpha(r)=0$, $dr+\beta(r)=0$. On définit d'abord une forme invariante φ, quadratique en du, dv. Si E est le birapport invariant de l'homographie découpée sur le rayon (u,v) par les surfaces (α) et (β) issues du point variable N du rayon $(u+du, v+dv)$, φ est la partie principale de $\frac{1}{4}(\log E)^2$. Deux autres invariants θ_1, θ_2 sont définis et on établit les relations liant φ, θ_1, θ_2. *M. Decuyper* (Lille).

Mayer, O. Sur les surfaces réglées. IV. Interprétation de l'arc projectif et des invariants h et j. Ann. Sci. Univ. Jassy. Sect. I. **26**, 626–632 (1940).

[For part III see the same vol., 299–308 (1940); these Rev. 1, 270.] L'arc projectif d'une surface réglée a été défini par E. Čech [G. Fubini et E. Čech, Geometria Proiettiva Differenziale, t. 1, Bologna, 1926, p. 207]. L'auteur en cherche une interprétation géométrique de la façon suivante. Soient M un point de la génératrice $g(u)$, M' l'intersection du plan tangent en M avec la génératrice $g(u+du)$ et M_1 l'intersection du plan tangent en M' avec g. La correspondance entre M et M_1 est une homographie dont le birapport invariant est désigné par E. L'auteur trouve que l'arc projectif est la partie principale de $\{6(\epsilon-1)/(\epsilon+1)\}^{\frac{1}{2}}$; cependant après avoir repris les calculs indiqués nous avons abouti à une expression un peu différente $\{(6-j)(\epsilon-1)/(\epsilon+1)\}^{\frac{1}{2}}$, où j est un invariant dont le sens est indiqué par le mémoire. L'auteur donne encore l'interprétation des invariants j et h définis dans le traité de Fubini et Čech. *M. Decuyper* (Lille).

Mayer, O. Sur les surfaces réglées. V. Les extrémales de l'arc projectif. Ann. Sci. Univ. Jassy. Sect. I. **27**, 3–11 (1941).

[Voir l'analyse ci-dessus.] L'auteur reprend l'expression de l'arc projectif σ d'une surface réglée qui, avec les notations de Fubini et Čech, est $d\sigma=|B^2-AC|^{\frac{1}{4}}du$, et il détermine les extrémales de l'arc projectif. Il fait une discussion complète des divers cas qui peuvent se présenter et termine en donnant les équations des surfaces réglées extrémales dans ces divers cas. *M. Decuyper* (Lille).

Rollero, A. Sulla determinazione del riferimento per lo studio proiettivo-differenziale delle superficie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 1059–1064 (1946).

The author presents an alternative to Bompiani's construction of an intrinsic reference system of the fourth order at a point of a surface in projective space [same Rend. (6) **25**, 149–154 (1937)]. In this system the surface takes the form $z=axy+\frac{1}{3}(x^3+y^3)+(Px+Ty)(x^3+y^3)/4!+\cdots$; P and T are invariants of which the writer gives new cross-ratio interpretations. The derivation of the reference system reveals an invariant curve on the surface containing all biflecnodal points. A modification of the method applied at a parabolic point leads to an intrinsic reference system in which the surface has the development of Popa, $z=y^2+x^3+x^4+a_{13}xy^3+\cdots$, where a_{13} is an invariant. *J. L. Vanderslice* (College Park, Md.).

Lalan, Victor. Sur un système de Pfaff de trois équations équivalent aux équations de Codazzi et de Gauss. C. R. Acad. Sci. Paris **224**, 518–520 (1947).

By choosing a convenient form for the first fundamental form of a surface in Euclidean three-space, namely $ds^2 = 2\omega_1\omega_2/A$, the author obtains a Pfaffian system equivalent to the equations of Gauss and Codazzi. This system consists of three equations, each of the last two being the prolongation of the previous equation in the sense of the theory of differential systems in involution.

C. B. Allendoerfer (Haverford, Pa.).

Masini Venturelli, Lucia. Sopra i ds^2 di Liouville di classe uno. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **102**, 145–163 (1943).

Masini Venturelli, Lucia. Le ipersuperficie di rotazione del tipo di Liouville. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **102**, 323–327 (1943).

[First paper.] If the linear element of a V_n $(n>2)$ of class one is $ds^2 = \sum_1^n U_i(u_i) \sum_1^n du_i^2$ then the direct computation (of the necessary and sufficient conditions for V_n to be of class one) gives

$$(1) \qquad ds^2 = \left(a^2 - b^2 \sum_1^n u_i^2\right) \sum_1^n du_i^2,$$

where a and b are constants and $\sum_1^n u_i^2 < \frac{3}{4} a^2/b^2$. [Second paper.] If, in particular, we are dealing with a hypersurface V_4: $x_1 = u \sin v_2$, $x_2 = u \cos v_2 \sin v_3$, $x_3 = u \cos v_2 \cos v_3 \sin v_4$, $x_4 = u \cos v_2 \cos v_3 \cos v_4$, $x_5 = f(u)$, then a suitable choice of parameter set and of $f(u)$ gives the linear element in the form (1).

V. Hlavatý (Prague).

Haimovici, Mendel. Sur les espaces d'Einstein à connexion affine. C. R. Acad. Sci. Paris **224**, 94–96 (1947).

L'auteur étudie les espaces introduits par Einstein et Bargmann [Ann. of Math. (2) **45**, 1–14 (1944); ces Rev. **5**, 218] comme espaces à connexion affine de Cartan, l'élément générateur étant un couple de points. Il explicite la connexion de Cartan de ces espaces (qui diffère de la connexion d'Einstein-Bargmann) et forme leurs équations de structure. Enfin il étudie les rapports géométriques qui peuvent exister entre la connexion ainsi obtenue et la connexion affine classique à parallélisme absolu ou celle de Cartan-Schouten.

A. Lichnerowicz (Strasbourg).

Wagner, V. Constant fields of local geometric objects in compound manifolds with a linear connexion. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 183–186 (1946).

The author's summary is as follows. As shown in the previous paper [same C. R. (N.S.) **40**, 94–97 (1943); these Rev. **6**, 106], the classification of connexions in the compound manifold $X_{n+(m)}$ according to their groups of holonomicity can be reduced to the finding of constant fields of local geometric objects in $X_{n+(m)}$. In the present note we shall give a general method for the solution of this problem.

S. Chern (Shanghai).

Vagner, V. The geometry of an $(n-1)$-dimensional nonholonomic manifold in an n-dimensional space. Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu] **5**, 173–225 (1941). (Russian) [MF 15611]

The theory of a nonholonomic variety imbedded in a space of one higher dimension is simpler than in the general case for several reasons, one of which is the possibility of keeping in closer touch with the problem of Pfaff; in this paper, which is partly expository in character, the author brings about additional simplification by introducing special systems of coordinates. After introducing and interpreting geometrically a tensor whose vanishing means that the variety is holonomic, he treats in detail many problems analogous to problems solved for holonomic varieties. He deals with linear connections, in particular those which preserve local volume, projective change of connection, varieties of zero projective curvature; he introduces a metric into the nonholonomic variety and discusses its relation with the metric of the imbedding space; he finds conditions for two metrics to be conformal and studies conformal connections. Finally, following J. Dubnov, he treats covariant integration in a V_3^2.

G. Y. Rainich (Ann Arbor, Mich.).

Vagner, V. Theory of congruences of circles in the geometry of a nonholonomic V_3^2 in R_3. Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu] **5**, 271–283 (1941). (Russian) [MF 15615]

Using simultaneously notations of vector calculus (including linear vector functions) and those of the Ricci calculus the author first studies a congruence of circles in Euclidean three-space. He uses in general the procedure and the results of the study of line congruences made by J. Dubnov [same Abh. **1**, 223–303 (1933)]. Among other things he finds necessary and sufficient conditions for a congruence of circles to be normal. In the case when it is not normal it determines a nonholonomic V_3^2 orthogonal to it. For this variety the author defines plane normal sections and introduces the curvatures of such sections as normal curvatures of the variety; these curvatures are expressed in terms of a certain linear vector function analogous to the one which is connected with the second differential form in the theory of surfaces. Conditions are given under which all points of a V_3^2 orthogonal to a congruence of circles are umbilical. Finally conditions are given for the possibility of "parallel imbedding" of a congruence of circles into a nonholonomic variety of zero curvature.

G. Y. Rainich.

Bompiani, E. Le connessioni tensoriali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 478–482 (1946).

Sur l'espace vectoriel $E_n^{(q)}$ des tenseurs contravariants d'ordre q définis sur un espace vectoriel E_n, l'auteur introduit des connexions qui peuvent ne pas se déduire par composition de connexions définies sur E_n. A ces connexions, il donne le nom de connexions tensorielles, par opposition avec celles définies sur l'espace vectoriel initial qui sont dites connexions vectorielles. Les notions classiques de tenseur de torsion relatif à une connexion, de différentielle absolu d'un champ de tenseurs dans une connexion sont étendues à ces connexions tensorielles. Une condition nécessaire et suffisante pour qu'une connexion tensorielle puisse se déduire d'une connexion vectorielle est obtenue.

A. Lichnerowicz (Strasbourg).

Bompiani, E. Connessioni del secondo ordine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 483–485 (1946).

Les ξ^i étant les composantes d'un vecteur, l'auteur cherche à quelles conditions une combinaison linéaire du type

$$\xi^i_{ml} = \partial_{ml}\xi^i + C^{i,\,q}_{ml,\,p}\partial_q\xi^p + D^i_{ml,\,p}\xi^p$$

présente le caractère tensoriel. Les coefficients C et D introduits définissent alors une "connexion du second ordre." La connexion du second ordre la plus générale s'exprime à partir d'une connexion affine ordinaire au moyen de deux tenseurs arbitraires. Si S^i_{pm} définit une connexion affine symétrique, les formules

$$C^{i,\,q}_{ml,\,p}=S^i_{pm}\delta^q_l+S^i_{pl}\delta^q_m-S^q_{ml}\delta^i_p,\quad D^i_{ml,\,p}=\partial_p S^i_{ml}$$

définissent une connexion du second ordre particulièrement simple. Celle-ci est étudiée dans le cas où la connexion S^i_{pm} est une connexion riemannienne. La dérivée riemannienne covariante seconde est alors donnée par

$$\nabla_{ml}\xi^i=\xi^i+R^i_{pm,\,l}\xi^p$$

et une formule analogue est valable pour l'extension second de ξ^i. *A. Lichnerowicz* (Strasbourg).

Théodoresco, N. Équations aux dérivées partielles et objets géométriques. Disquisit. Math. Phys. 4, 105–116 (1945).

A completely linear partial differential equation of the second order has been written in invariant form by various authors by the use of a metric of Riemann, Weyl or Finsler type. The present author attacks the third order case, regarding his procedure as an exercise in the investigation and calculation of new "geometric objects" (as defined, for example, by Veblen) and as an example of the extension of tensor calculus in a new direction. The given equation is

$$a^{ijk}\frac{\partial^3 f}{\partial x^i\partial x^j\partial x^k}+b^{ij}\frac{\partial^2 f}{\partial x^i\partial x^j}+c^i\frac{\partial f}{\partial x^i}+af=0.$$

The transformations involved are general point transformations on the coordinates x^i. The final invariant form, at its simplest, is

$$a^{ijk}f_{,ijk}+a^{ij}f_{,ij}+a^if_{,i}+af=0,$$

where a comma denotes covariant differentiation with respect to an assumed affine connection and a^{ij}, a^i (as well as a^{ijk}) are tensors explicitly given as functions of the original coefficients, the affine connection and the latter's first derivatives. The most general invariant form is also exhibited. It involves, besides the above quantities, seven tensors arbitrary except for certain symmetry and contraction conditions. *J. L. Vanderslice* (College Park, Md.).

***Bortolotti, Enea. Geometria di sistemi alle derivate parziali.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 323–337. Edizioni Cremonense, Rome, 1942.

Salvetti, Carlo. Numeri di Clifford e operatori nello spazio a cinque dimensioni. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 9(78), 347–359 (1945).

A vector in five-dimensional space is defined by

$$v_1\Gamma_1+v_2\Gamma_2+v_3\Gamma_3+v_4\Gamma_4+v_5\Gamma_5,$$

where the Γ's are numbers of Clifford satisfying the conditions $\Gamma_j\Gamma_k+\Gamma_k\Gamma_j=2\delta_{jk}$. This allows the formation of alternating tensors (bivectors, trivectors and quadrivectors) and it is shown how, with the aid of these numbers, the notions of ordinary vector analysis can be generalized.

D. J. Struik (Cambridge, Mass.).

Caldonazzo, Bruto. Meccanica. Sugli invarianti cubici propri di un tensore doppio. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 8(77), 24–30 (1944).

The author applies the method of vectorial homography to discuss the cubic invariants of a Cartesian tensor of the second order in three dimensions [cf. U. Cisotti, third reference in the following review]. He obtains homographic expressions for the two proper cubic invariants and shows that they vanish when the tensor is skew-symmetric. Some applications are given. *J. L. Synge* (Pittsburgh, Pa.).

Cisotti, Umberto. Invarianti di vettori e di tensori doppi. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 8(77), 253–258 (1944).

For a 3-space it is shown that a Cartesian vector T_i admits only one invariant of even degree $2m$, namely $(T_iT_i)^m$, and no semi-invariant, a semi-invariant being a quantity invariant under rotation of axes but changing sign on reflection of axes [cf. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 1, 337–341 (1940); these Rev. 1, 325]. The author also considers invariants of degree m formed from a tensor of the second order T_{ik}, the general formula agreeing with results obtained in earlier papers for the cases $m=2$ and $m=3$ [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 511–516 (1941); 4, 8–11 (1943); these Rev. 8, 232, 233]. Quartic invariants of a tensor T_{ik} are also considered; 24 such invariants are found, but of these only 7 are distinct; of these 7, 5 are called improper because they are powers and products of linear, quadratic or cubic invariants; the remaining 2 proper invariants are $T_{qp}T_{pr}T_{sq}T_{rs}$ and $T_{sp}T_{pq}T_{qr}T_{rs}$. *J. L. Synge* (Pittsburgh, Pa.).

Cisotti, Umberto. Tensore isotropo o emisotropo di minimo scarto da un tensore assegnato. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 4(73), 85–93 (1940).

This paper deals with Cartesian tensors in 3-space; its purpose may be briefly described as follows: to find the isotropic tensor which approximates most closely to a given tensor of even order and to find the skew-isotropic tensor which approximates most closely to a given tensor of odd order. Let $T_{i_1\ldots i_mk_1\ldots k_m}$ be a given tensor of even order, and $\Delta^{(r)}_{i_1\ldots i_mk_1\ldots k_m}$ $(r=1,\cdots,N)$ the N isotropic tensors of the same order, these latter being the set of all distinct expressions of the form $\delta_{i_1k_1}\delta_{i_2k_2}\cdots\delta_{i_mk_m}$. Any linear combination of the Δ's is an isotropic tensor. The problem is to find scalars $A_1,\cdots,A_N$ so as to minimize the quantity

$$S^2=\Big(T_{i_1\ldots k_m}-\sum_{r=1}^{N}A_r\Delta^{(r)}_{i_1\ldots k_m}\Big)\Big(T_{i_1\ldots k_m}-\sum_{s=1}^{N}A_s\Delta^{(s)}_{i_1\ldots k_m}\Big).$$

It is shown that the A's must satisfy

$$(*)\qquad \sum_{r=1}^{N}3^{N_{rs}}A_r=T_{i_1\ldots k_m}\Delta^{(s)}_{i_1\ldots k_m},\qquad s=1,\cdots,N,$$

where N_{rs} are positive integers given by

$$3^{N_{rs}}=\Delta^{(r)}_{i_1\ldots k_m}\Delta^{(s)}_{i_1\ldots k_m}.$$

The determinant of the equations $(*)$ is positive. For $m=1$, the given tensor is T_{ik}; the isotropic tensor minimizing S^2 is $\frac{1}{3}T_{jj}\delta_{ik}$. Details are also given for $m=2$. If the given tensor is of odd order, the argument proceeds along similar lines, the isotropic Δ's being replaced by the skew-isotropic tensors $\epsilon_{i_1i_2k}\delta_{i_3k_1}\cdots\delta_{i_mk_m}$. *J. L. Synge* (Pittsburgh, Pa.).

NUMERICAL AND GRAPHICAL METHODS

***Tables of Spherical Bessel Functions.** Prepared by the Mathematical Tables Project, National Bureau of Standards. Vol. I. Columbia University Press, New York, 1947. xxviii+377 pp. $7.50.

The purpose of the tables is described in a foreword by P. M. Morse. "There are only eleven . . . coordinate systems in which the wave equation can be separated. . . . In the solutions for six of these . . . the ubiquitous Bessel functions are involved. . . . In four systems . . . Bessel functions are involved having orders equal to one-half an odd integer. Curiously enough, satisfactory tables of these important functions have not heretofore been available. The publication of the present tables thus fills a long-felt need." Beside wave problems, the tabulated functions arise also in potential problems and in heat conduction.

An introduction by A. N. Lowan contains a list of the principal formulae, an account of the method of computation, preparation of the manuscript and interpolation in the tables. The tabular values were obtained from "key values" either by the "derivative method" [described in Tables of Sine, Cosine and Exponential Integrals, vol. II, New York, 1940, pp. xiii–xiv; these Rev. **2**, 366] or by the use of recurrence relations. Three methods were used for the computation of the "key values" themselves: (i) the power series, (ii) the asymptotic series (which terminate when the order is half an odd integer) and (iii) the τ-method which is based on H. Jeffreys' approximation of solutions of linear differential equations, has been developed by C. Lánczos and W. Horenstein and is described in detail in the introduction. In essence, the τ-method is an approximation to $J_\nu(x)$, when 4ν is close to 125 and x ranges between 0 and 25, in the form

$$\frac{(x/2)^\nu}{\Gamma(\nu+1)}e^{-x^2/125}g(x^2/625),$$

where g is a polynomial of the ninth degree. The coefficients of this polynomial and an estimate of the error are given.

The tabular material is as follows. Tables of the function $(\frac{1}{2}\pi/x)^{\frac{1}{2}}J_\nu$ for $\pm\nu=0.5(1.0)12.5$ and $x=0(0.01)10(0.1)25.0$; and for $\pm\nu=13.5$ and $x=0(0.01)10.00(0.05)10.50(0.1)25.0$. Tables of $x^{-\nu}(\frac{1}{2}\pi)^{\frac{1}{2}}J_\nu(x)$ for $\nu=-0.5$ and for $\pm\nu=1.5(1.0)13.5$ and $x=0(0.01)x_\nu$, where $x_\nu=0.50$ for $\pm\nu=1.5$; $x_\nu=1.00$ for $\nu=-0.5, \pm2.5, \pm3.5$; $x_\nu=1.50$ for $4.5\leqq\pm\nu\leqq7.5$; $x_\nu=2.00$ for $8.5\leqq\pm\nu\leqq11.5$; and $x_\nu=2.50$ for $\pm\nu=12.5, 13.5$. Tables of interpolation coefficients.

The majority of the entries are given to eight significant figures for $x<10$ and to seven significant figures for $x>10$, except near the zeros of the functions. Second central differences (sometimes modified) are tabulated alongside the entries, except for regions close to the origin of x. In a few places fourth central differences are also given. The production of the tables maintains the high standard of former MTP tables. *A. Erdélyi* (Pasadena, Calif.).

***Tables of the Bessel Functions of the First Kind of Orders Zero and One, by the Staff of the Computation Laboratory.** The Annals of the Computation Laboratory of Harvard University, vol. III. Harvard University Press, Cambridge, Mass., 1947. xxxvii+652 pp. $10.00.

***Tables of the Bessel Functions of the First Kind of Orders Two and Three, by the Staff of the Computation Laboratory.** The Annals of the Computation Laboratory of Harvard University, vol. IV. Harvard University Press, Cambridge, Mass., 1947. v+652 pp. $10.00.

These are the first two volumes of a set of tables of $J_n(x)$ for all integer n from 0 to 100 and values of x between 0 and 25 at intervals of 0.001 and between 25 and 100 at intervals of 0.01. The very size of this gigantic project makes one sit up and realise the change in the whole outlook in scientific computing brought about by the employment of modern machinery. And so does the spirit in which the job was tackled. When it became clear that for the computation of $J_{100}(x)$ to ten decimal places an adding, multiplying and storage capacity of more than forty digits would be required, the authors cheerfully say that "it was only necessary to link two normal adding storage registers each comprising twenty-three digits and the algebraic sign to form a single adding storage register covering forty-six digits and the algebraic sign." Such extensive tables of Bessel functions will be useful in many problems of physics and engineering. Among such problems, frequency modulation, resonance in cavities, waves in various media and the vibration theory of structures are mentioned in the introduction. The work was carried out by the staff of the Computation Laboratory, on the Automatic Sequence Controlled Calculator. The project was under the direction of H. H. Aiken, the required circuits and all their wiring was the work of R. L. Hawkins and the control tapes were designed by R. M. Bloch.

The power series were used to evaluate $J_n(x)$ for $n=0, 1, 2, 3$ and $x=0(0.001)2.000$, and $n=0, 1, 2$ and $x=2.00(0.01)25.00$. In the latter range the functions were subtabulated by means of the Newton-Bessel central difference interpolation formula. The asymptotic expansion was used to obtain the function in question for $n=0, 1, 2$ and $x=25.00(0.01)99.99$. All other functions were obtained from the recurrence relations satisfied by Bessel functions. The results were checked automatically and printed by typewriters controlled by the machine itself. In the published volumes the original pages are reproduced by means of offset lithography. The printed tables were read against several published tables; no discrepancy was observed. Sixth differences (for $x<25$) or ninth differences (for $x>25$) were continuously recorded by the calculator by means of a second typewriter. The smooth behaviour of these differences together with other criteria indicate that the twenty-three decimal values of $J_0(x)$ to $J_3(x)$ are correct to within three units of the last decimal place.

The first volume contains a preface (by Aiken) and an introduction (by Bloch) recalling the basic formulae, describing the computation of the tables and giving instructions for interpolation. The following tables are given: the first 61 coefficients in the power series of $J_0(x)$, $J_1(x)$, $J_2(x)$, to 50 significant figures; the first 16 coefficients in the power series of $J_3(x)$ to 40 significant figures; the first 46 coefficients in the asymptotic expansions of $J_0(x)$, $J_1(x)$, $J_2(x)$, to 30 significant figures. In all tables of coefficients marginal entries indicate the range of x throughout which the term containing the coefficient in question was the last to be retained. The principal table, forming the bulk of the volume, is the table of $J_0(x)$ and $J_1(x)$ to 18 decimal places for $x=0(0.001)25.000(0.01)99.99$. No differences are given. The second volume contains the corresponding tables of $J_2(x)$ and $J_3(x)$.

The great importance of systematic basic tables of Bessel functions covering a large range of both order and variable is too obvious to need any elaboration here. It is perhaps not too much to say that a project of this magnitude is epoch-making in scientific computing. *A. Erdélyi.*

Durañona y Vedia, Agustín. Calculation of $\sqrt{N}$ by means of a generalized continued fraction. Math. Notae 6, 116–118 (1946). (Spanish)

Assuming $N>1$ and starting with any estimate α for the value of $\sqrt{N}$, subject only to the condition $\alpha^2<N<5\alpha^2$, the author sets up a continued fraction for $\sqrt{N}$, proves its convergence and gives an estimate of the error of the nth convergent. *W. E. Milne* (Corvallis, Ore.).

***Malmquist, F. Approximation of real functions by linear exponential expressions.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 70–76. Jul. Gjellerups Forlag, Copenhagen, 1947.

The problem considered is that of approximating a given real function $f(x)$ by an expression of the form $g(x)=\sum_{i=0}^{n}A_ie^{a_ix}$. Solutions are obtained for two methods of obtaining rough preliminary values of the parameters: viz., (1) such that the values of the ordinates and the first $2n-1$ derivatives of $f(x)$ and $g(x)$ are equal for a specified argument, and (2) such that $f(x)$ and $g(x)$ have the same values for $2n$ equally spaced arguments. *T. N. E. Greville.*

Mikheladze, Š. On a process of interpolation for functions of two variables. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 6, 503–509 (1945). (Georgian. Russian summary)

The author presents the usual two-variable interpolation formula in terms of differences for a right triangular array of points in the (x, y)-plane. The formula is given in four different forms corresponding to the four possible orientations of the right triangle, i.e., right angle in the N.W., N.E., S.W. or S.E. corner. *W. E. Milne* (Corvallis, Ore.).

Bruner, Nancy. Note on the Doolittle solution. Econometrica 15, 43–44 (1947).

In applying the Doolittle solution of simultaneous equations in correlation the author recommends that first of all the equations be rearranged so that the elements of the principal diagonal increase (or at least do not decrease) in magnitude in going from upper left to lower right. The reason for this is that the check column then gives a closer check, being less affected by rounding errors.

W. E. Milne (Corvallis, Ore.).

Leavens, Dickson H. Accuracy in the Doolittle solution. Econometrica 15, 45–50 (1947).

The author investigates further the problem of accuracy treated in the paper reviewed above and discusses "the relative accuracy and convenience of four methods of applying the Doolittle method, as follows: Method A. Straightforward solution of the equations with the data of given magnitude and in given order. Method B. Solution with the data rearranged so that the terms in the principal diagonal increase in size. Method C. Solution with the data in original order but adjusted so that the elements of the principal diagonal range between 0.1 and 10.0. Method D. Solution with the data adjusted to represent coefficients of correlation between pairs of variables, thus making all the elements of the principal diagonal unity."

His rather cautious conclusion is that it would "seem to be good practice to use Method B rather than Method A whenever the original data vary much in size, provided the order in which the equations are arranged is immaterial and provided the extra information obtainable from Method D is not wanted. In cases where it is desired to test the effects of adding one variable at a time Method C may be useful."

W. E. Milne (Corvallis, Ore.).

Hughes, Richard H., and Wilson, E. Bright, Jr. An electrical network for the solution of secular equations. Rev. Sci. Instruments 18, 103–108 (1947).

The characteristic values of a real symmetric matrix A can be obtained by an electrical network which finds the n roots of the determinantal equation $|A-xI|=0$. The network consists of n junction points, each connected to every other by fixed reactive admittances (condensers or induction coils) equal to the off-diagonal terms of the determinant and each connected to ground through a fixed admittance equal to the negative of the sum of the constant part of the diagonal term and the off-diagonal terms in the same row or column. In addition, each junction is connected to ground through a variable admittance which represents the unknown in the equation; these latter admittances are equal for all the junction points. While the network is excited with alternating current from a constant current source of constant frequency, the variable admittances are all altered simultaneously by the same amount until all the maxima of the electromotive forces of the junction points with respect to ground have been found. The values of the variable admittances at the several maxima constitute the desired roots. By the use of commercially available radio parts, an accuracy of one percent is obtained. Previous attempts at solving the problem by a direct simulation with electrical networks and varying the frequency were not successful because the admittances varied with the frequency. *M. Goldberg* (Washington, D. C.).

Holzer, L., und Melan, E. Ein Beitrag zur Auflösung linearer Gleichungssysteme mit positiv definiter Matrix mittels Iteration. Akad. Wiss. Wien, S.-B. IIa. 151, 249–254 (1942).

The paper deals with the convergence of the process of iteration applied to the solution of a system of algebraic equations with matrix A. For the case in which A is the symmetric matrix of a positive definite quadratic form so that its characteristic roots are real and positive, it is possible to put the original equations in a form for which the process of iteration will converge. The practical application of this procedure requires the knowledge of an upper bound for the characteristic roots of A. Six different inequalities are offered for the determination of an upper bound. *W. E. Milne* (Corvallis, Ore.).

Bodewig, E. Comparison of some direct methods for computing determinants and inverse matrices. Nederl. Akad. Wetensch., Proc. 50, 49–57 (1947).

The purpose of this paper is first to decide the question of priority in certain recent methods for computing determinants and inverse matrices, the methods in question being typified by that of Banachiewicz. According to the author the basic formula of these methods was published by Schur in 1917 and was rediscovered by Banachiewicz in 1937. The second purpose of the paper is to compare the actual labor of computation by the standard method of successive elimination with that for methods of the Banachiewicz type. The author compares the total number of multiplications and additions and concludes that the standard method is best for linear equations and determinants, the Banachiewicz method best for inverse matrices. *W. E. Milne.*

Labrouste, H., et Labrouste, Y. Analyse des graphiques résultant de la superposition de sinusoïdes. Mém. Acad. Sci. Inst. France (2) 64, no. 5, 84 pp. (1941).

Les auteurs développent en détail leur méthode qui consiste à supprimer toutes les composantes de la fonction, à l'exception d'une seule, par une combinaison linéaire des valeurs de la fonction. Supposons la fonction y:

$$y=a \sin (\theta x+\phi)+a_1 \sin (\theta_1 x+\phi_1)+a_2 \sin (\theta_2 x+\phi_2)+\cdots, \quad \theta_i=2\pi/n_i.$$

Alors la méthode est fondée sur le fait que la somme $Y_m=y_m+y_{-m}=A \sin (\theta x+\phi)+A_1 \sin (\theta_1 x+\phi_1)+\cdots$, où $A_i=2a_i \cos m\theta_i$, $y_{\pm m}=y(x_0\pm m)$ et où x_0 parcourt tous les arguments donnés, est une transformation nondéphasante, tandis que les amplitudes ont les facteurs $\alpha_m=2\cos m\theta_i$. Toute combinaison linéaire des Y_m a des propriétés semblables et tout revient à l'étude des facteurs d'amplitude en fonction de m et n. Les auteurs font usage principalement des combinaisons suivantes: (a) $\pi(Y)=Y_m \cdots Y_r$ et $\pi(s)=s_m \cdots s_r$ ayant les facteurs $\pi(\alpha)=\alpha_m \cdots \alpha_r$ et $\pi(\sigma)=\sigma_m \cdots \sigma_r$, où $\alpha_m=2\cos 2\pi m/n$, $s_m=y_0+Y_1+\cdots+Y_m$, $\sigma_m=1+\alpha_1+\cdots+\alpha_m$; (b) $\pi(Z)=Z_m \cdots Z_r$ en nombre pair et $\pi(T)=T_m \cdots T_r$ en nombre pair avec les facteurs $\pi(\beta)=\beta_m \cdots \beta_r$ et $\pi(\tau)=\tau_m \cdots \tau_r$, où $Z_m=\pm(y_m-y_{-m})$, $\beta_m=2 \sin 2\pi m/n$, $T_m=Z_1+\cdots+Z_m$, $\tau_m=\beta_1+\cdots+\beta_m$ (Z_m est la seule transformation déphasante employée); (c) des combinaisons mixtes d'une combinaison (a) avec (b); (d) une combinaison (a)–(c) combinée avec une Y ou Z; (e) $\pi(s_N)=(s_m)_N \cdots (s_r)_N$, $\pi(s_{N/2})=(s_m)_{N/2} \cdots (s_r)_{N/2}$, $\pi(T_{N/4})=(T_m)_{N/4} \cdots (T_r)_{N/4}$ avec les facteurs correspondants, où $(s_m)_N=y_0+Y_N+Y_{2N}+\cdots+Y_{mN}$, $(s_m)_{N/2}=y_0-Y_{N/2}+Y_N-\cdots+(-1)^m Y_{mN/2}$, $(T_m)_{N/4}=Z_{N/4}-Z_{3N/4}+\cdots\pm Z_{mN/4}$. Chacune de ces combinaisons équivaut à une combinaison $\sum K_i Y_i$ linéaire. Son efficacité dépend du nombre de termes Y.

Les "courbes de sélectivité" des transformations (a)–(d) (c'est-à-dire, les courbes $\pi(\alpha)$, $\pi(\sigma)$, $\cdots$ en fonction de n) ont toutes quelques zéros au commencement et se séparent alors en deux types. (a') La courbe a une "boucle ouverte," c'est-à-dire, ayant une valeur non-nulle pour $n=\infty$. Ce sont les combinaisons (a). Elles suppriment donc les composantes de petites périodes inférieures à n' et laissent subsister les autres. (b') Une "boucle fermée." C'est-à-dire elles ont une valeur nulle pour $n=\infty$. Ce sont les combinaisons (b) et (c), tandis que (d) a encore un zéro entre l'abscisse de la boucle et l'infini. Elles suppriment les composantes de petites et de grandes périodes et laissent subsister celles de périodes moyennes d'un intervalle limité. Les courbes (e) ont une boucle très aigue pour $n=N$ et les harmoniques isolent donc une composante dont la période est connue approximativement.

En pratique on combine les opérations mentionnées. Les calculs peuvent être simplifiés par des simples relations. Un procédé semblable s'applique aux composantes d'amplitude variable, $y=f(x) \sin (\theta x+\phi)$. Les lois linéaire et exponentielle d'amplitude sont traitées en détail.

En pratique on procède comme il suit. On commence par une solution approximative en appliquant différents $\pi(s)$ jusqu'à ce que la période la plus grande soit isolée et en traitant alors le reste (après soustraction) de la même manière. Alors on isole chaque composante trouvée approximativement par une combinaison (c) ou (d). Un tableau en facilite le choix aussi que des tables numériques et des courbes [voir l'analyse suivante]. Le mémoire finit par une comparaison de la méthode aux autres procédés de calcul (moyennes, Fourier). La méthode des auteurs est plus générale et plus effective. *E. Bodewig* (La Haye).

✶Labrouste, H., et Labrouste, Y. Analyse des Graphiques Résultant de la Superposition de Sinusoïdes. Tables Numériques Précédées d'un Exposé de la Méthode d'Analyse par Combinaisons Linéaires d'Ordonnées. Presses Universitaires de France, Paris, 1943. iii+204 pp.

✶Labrouste, H., et Labrouste, Y. Atlas de Courbes de Sélectivité. Supplément aux Tables Numériques pour l'Analyse des Graphiques Résultant de la Superposition de Sinusoïdes. Presses Universitaires de France, Paris, 1943. 1 p.+35 plates in a portfolio.

L'ouvrage comprend trois parties. (1) Une reproduction du mémoire analysé ci-dessus. (2) Des tables numériques destinées à faciliter les calculs pour connaître d'avance le résultat à attendre de l'emploi de telle ou telle combinaison. Elles servent aussi au calcul définitif de l'amplitude des composantes trouvées. Elles donnent, en fonction de n, les valeurs des facteurs d'amplitude α_m et β_m des combinaisons simples Y_m et Z_m et celles des facteurs d'amplitudes σ_m, $(\sigma_m)_{N/2}$ et τ_m des combinaisons complexes. Les coefficients d'amplitude des combinaisons multiples se déterminent ou par l'usage répétée des tables numériques ou par les courbes mentionnées ci-dessous. Les tables s'étendent de $n=0.5$ jusqu'à $n=100$ par des intervalles variants et sont données avec 3 décimales. (3) Un atlas. Celui-ci contient les courbes de sélectivité de diverses combinaisons multiples et des tables numériques des K_ν qui décomposent les mêmes combinaisons en éléments simples $\sum K_\nu Y_\nu$. Pour les combinaisons plus compliquées il faut combiner des résultats lus sur plusieures des courbes traitées. *E. Bodewig* (La Haye).

Feldman, M. R. Application of Galerkin's method to difference equations. Engineering Rev. [Akad. Nauk SSSR. Inženernyĭ Sbornik] 2, no. 1, 67–70 (1943). (Russian. English summary)

The author gives an approximate method of solving difference equations. With the aid of Galerkin's method he improves considerably on the method of the Ritz type previously used by Bleich and Melan. The author's scheme has the advantage of comparative simplicity and is effective, in particular, in the numerical work related to some types of differential equations (use of determinants of increasingly high orders is avoided). *W. Trjitzinsky* (Urbana, Ill.).

Shortley, George, Weller, Royal, Darby, Paul, and Gamble, Edward H. Numerical solution of axisymmetrical problems, with applications to electrostatics and torsion. J. Appl. Phys. 18, 116–129 (1947).

This gives an extension of the numerical methods for handling partial differential equations in two dimensions which were developed by Shortley, Weller and Fried [Ohio State University Studies, Engineering Ser., v. 11, no. 5. Engineering Experiment Station, Bull. no. 107, 1940; these Rev. 2, 368]. The authors' summary states: "Numerical methods are given for solution of axisymmetrical problems involving the partial differential equation $\psi_{zz}+\psi_{\rho\rho}+K\rho^{-1}\psi_\rho=0$, where ρ is the radial coordinate and z the coordinate parallel to the axis. The various values of K which occur in physical situations are discussed, and common iteration methods for handling these problems are given. For Laplace's equation $K=1$. For the Stokes stream function, $K=-1$, but it is pointed out that for numerical work a new function, called the flow-disturbance function, having $K=3$, is more tractable. Similarly a new function with $K=5$, the stress-concentration function, is much easier to compute than the usual stress-function ($K=-3$) for the

case of the torsion of a circular shaft of varying diameter. The methods are illustrated by computation of the equipotentials for an electron lens, and by a complete computation of the stresses and strains in a particular grooved circular shaft under torsion."

The numerical methods concerned involve iteration, or "relaxation" over a two-dimensional grid, by formulae obtained on replacing derivatives by appropriate differences. The paper lists formulae appropriate, for the equation concerned, to cases of unequal spacing of the points, to evaluations near the boundary and to points near which the spacing between points is doubled or halved. Tables are given, for $K=1$ and 5, for the improvement of the value at the centre point of a block of nine points, without using any of the old values at these points; this follows up the discussion in the earlier work on the increase of rapidity of convergence obtained by improving a block of several points all at once. *J. C. P. Miller* (London).

***Bergström, Harald. On some approximate solutions of the equation of conduction and their connection with distribution functions.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 271–280. Jul. Gjellerups Forlag, Copenhagen, 1947.

Apparently unaware of the extensive literature on the subject, the author re-examines the connection between the heat equation $u_t=u_{xy}$ and the difference equation

$$u_{k,n+1}=\tfrac{1}{2}\{u_{k-1,n}+u_{k+1,n}\}.$$

An estimate of the error is furnished by familiar facts concerning the normal (Gaussian) approximation to the binomial distribution. *W. Feller* (Ithaca, N. Y.).

Crank, J., and Nicolson, P. A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type. Proc. Cambridge Philos. Soc. **43**, 50–67 (1947).

Three approximate methods for the solution of the nonlinear equation of heat flow in a medium where heat is being generated by a chemical reaction are compared. The equations are $\theta_t=\theta_{xx}-qw_t$, $w_t=-kw\exp(-A/\theta)$, where subscripts indicate partial differentiations and q, k, A are constants. The boundary conditions are one-point with respect to time t: θ and w are given functions for $t=0$, $0<x<1$, and two-point with respect to x: θ_x is a given function for $x=0$ and for $x=1$, $t\geqq 0$.

The first method consists in reduction to a system of ordinary differential equations by approximating the time derivatives with differences; the second is the same except that the rôles of x and t are interchanged, i.e. the space derivatives are approximated by differences, giving a system of ordinary differential equations on t; the third method consists in approximating all the derivatives by differences. The authors conclude from experimentation with these procedures that the last is much faster and more satisfactory. The ordinary differential equations obtained by the first two methods were integrated with a differential analyser.

In the third method a system of nonlinear algebraic equations is obtained, which is solved by a combination iterative and step by step method. That is, the two point boundary conditions on x are satisfied by iteration and this whole process is carried forward step by step with respect to t. Several variations of this method are discussed, one of which avoids iteration entirely, but the iteration method is considered best. Application is made to the case of a wooden plank which is subjected to heat and flame along the edge and sides until combustion takes place.

P. W. Ketchum (Urbana, Ill.).

Evans, A. W. Further remarks on the relationship between the values of life annuities at different rates of interest, including a description of a method of first-difference interpolation and a reference to annuities-certain. J. Inst. Actuar. **72**, 447–454 (1946).

***Zwinggi, Ernst. About a form of representation of the policy value.** Försäkringsmatematiska Studier Tillägnade Filip Lundberg, pp. 286–292. Stockholm, 1946.

Application of the method previously described in Mitt. Verein. Schweiz. Versich.-Math. **45**, 375–383 (1945); Experientia **2**, 182–183 (1946); these Rev. **7**, 340; **8**, 58.

W. Feller (Ithaca, N. Y.).

ASTRONOMY

Iljinsky, I. A simple method for the determination of orbits of planets and comets. Astr. J. Soviet Union [Astr. Zhurnal] **23**, 367–376 (1946). (Russian. English summary)

The problem faced by a computer of the orbit of a newly discovered comet is to derive a precise enough ephemeris to enable observers to follow the comet. There are a large number of methods available for this purpose. Every computer has his own little short-cuts and simplifications. This paper is typical of many papers of similar contents. The simplification in this case is the introduction of a graphical solution in order to avoid solving Lagrange's equation of the eighth degree or the transcendental equation of Gauss of the fourth degree. The graphs refer to the ratio of the triangles made by the radii vectores of the body as functions of the heliocentric distance.

The Russian title is "The simplest method. . . ." From a study of the formulae involved plus the necessity of a very careful construction of several graphs, the method does not appear so simple. It is hardly likely that the labor involved in this procedure will be less than the computation by the orthodox analytical methods, especially using calculating machines. The example of computation given, that of the orbit of Comet 1882II, is not convincing. After two approximations the derived elements are still very far from the final ones, the difference in the node being 26′51″. The orbit obtained by R. Gautier from the same observations [Astr. Nachr. **105**, 363–366 (1883)] by actual calculations was very much nearer the truth. *N. T. Bobrovnikoff.*

Roure, Henri. Théorie nouvelle des planètes du système solaire. C. R. Acad. Sci. Paris **223**, 885–887 (1946).

The author proposes to use for the development of planetary theories the equations and variables that were used in the Hill-Brown lunar theory. [Only a brief indication of the procedure is given, insufficient to permit an estimate of the merits of the method.] *D. Brouwer.*

Armellini, Giuseppe. Il problema ristretto lineare dei tre corpi. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 15–22 (1941).

The author considers a restricted three-body problem in which two finite masses move on a fixed straight line in a

particular solution of the two-body problem for which the energy constant is zero, while an infinitesimal mass P moves on the same line subject to the attraction of the two finite masses. It is shown that the differential equations for P can be reduced to a first order equation of a type studied by Abel. *W. Kaplan* (Ann Arbor, Mich.).

Armellini, Giuseppe. Contributo alla dinamica del sistema galattico. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 73–82 (1941).

In order to explain the observed distribution of star velocities, the author proposes a distribution function based on three integrals of the equations of motion: the energy integral, the angular momentum integral and an approximate integral based on the position of the sun in a plane of symmetry of the galaxy. It is shown that a suitable combination of these integrals leads to an ellipsoidal velocity distribution in accordance with observations. [Similar ideas are proposed and carried through in detail by S. Chandrasekhar, Principles of Stellar Dynamics, University of Chicago Press, 1942, especially pp. 86 ff.; these Rev. **4**, 57.] *W. Kaplan* (Ann Arbor, Mich.).

Armellini, Giuseppe. I problemi fondamentali della cosmogonia e la legge di Newton. V. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 121–126 (1940).

Armellini, Giuseppe. I problemi fondamentali della cosmogonia e la legge di Newton. VI. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 697–704 (1940).

Armellini, Giuseppe. I problemi fondamentali della cosmogonia e la legge di Newton. VII. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 158–165 (1940).

Armellini, Giuseppe. I problemi fondamentali della cosmogonia e la legge di Newton. VIII. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 302–311 (1941).

Armellini, Giuseppe. I problemi fondamentali della cosmogonia e la legge di Newton nel caso di più pianeti. IX. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 748–754 (1942).

Armellini, Giuseppe. Sopra l'età dei pianeti e sopra l'incremento dei parametri delle loro orbite, a causa del termine cosmogonico. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 229–235 (1942).

These papers continue previous researches of the author [cf. part IV, Atti Accad. Naz. Lincei. Rend. (6) **29**, 649–655 (1939); these Rev. **1**, 61; cf. also Chazy, C. R. Acad. Sci. Paris **210**, 713–716 (1940); these Rev. **2**, 264] on the implication of the addition of a term proportional to dr/dt to the Newtonian law of attraction, r being the mutual distance. In part V it is shown that the theory predicts a gradual diminution of the solar angular momentum and a corresponding increase of that of the planets. In part VI it is shown that the gradual expansion of the solar system predicted by the theory leads to a reasonable value of the "age" of the solar system (assumed formed by explosion of the sun) and that the values of eccentricities, inclinations and angular momenta of the planetary orbits are qualitatively correctly predicted. In part VII the theory is applied to a general n-body problem and it is shown that, if the system remains bounded and there are no collisions as t becomes infinite, then either the system asymptotically approaches a planar rigid body motion or else the system "condenses" on its center of mass. In part VIII the limiting case of a planar rigid body is assumed for the solar system and the radii of the resulting circular orbits about the sun are determined; each of these radii satisfies a cubic equation with two positive roots, whose significance is discussed. In part IX it is shown that the perturbation effects of the attraction of other planets lead to a more rapid rate of expansion of the solar system than would occur if these effects did not exist. In the final paper the question of the value of the basic constant of proportionality is reexamined and the possibility of its being matched to the age of the solar system and other observed quantities is discussed. *W. Kaplan* (Ann Arbor, Mich.).

Agostinelli, Cataldo. Sui problemi fondamentali della cosmogonia. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 166–186 (1940).

Agostinelli, Cataldo. Effetto del termine cosmogonico sullo spostamento del perielio di una orbita planetaria e sulla variazione del parametro e dell'eccentricità. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 592–602 (1941).

These papers continue the work of Armellini [cf. the preceding review]. The first is an extension of the theory to the solar system with no assumption made concerning the shape of the sun; it is shown that as t becomes infinite the mean motion of the planets and the angular velocity of the sun approach equality. In the second paper, it is shown that the hypothesis of the term in dr/dt in the gravitational law can explain the advance of the perihelion of Mercury. *W. Kaplan* (Ann Arbor, Mich.).

Fessenkoff, B. On the motion of meteoric particle in the interplanetary space. Astr. J. Soviet Union [Astr. Zhurnal] **23**, 353–366 (1946). (Russian. English summary)

The problem of the motion of a small particle under the combined action of gravitation and light pressure of the sun has engaged the attention of many investigators since 1900. A very thorough treatment of this problem is given by Seeliger [see, for instance, Astr. Nachr. **187**, 417–422 (1911)]. Fessenkoff's treatment does not differ greatly from that of Seeliger but Seeliger's name is not mentioned. The problem is essentially indeterminate, as the light pressure depends on the size of the particle and it is certain that particles of all sizes are present in interplanetary space. The result then can be given only in general terms.

The equations of motion can be written in the form $md^2x/dt^2 = X_1 + X_2$, where X_1 is the gravitational component $-k^2Mmx/r^3$ and X_2 is the light pressure. Since the velocity of the particle is finite in comparison with the velocity of light, the terms due to the aberration of light must also be included in the expression for X_2. These equations can be integrated in a number of ways, but since the primary interest is in the heliocentric distance of the particle and its period, only these two quantities are considered. Tables for a numerical integration of the complicated expressions are given. Generally speaking, radiation pressure acts as a brake on the motion of the particle, so that the particle will describe a spiral ending in the sun. Approximate lengths of time of the existence of the particle before it falls into the sun are evaluated for various initial conditions. Finally the possible influence of the magnetic and electric fields of the sun is considered, but is shown to be negligible. *N. T. Bobrovnikoff* (Delaware, Ohio).

★Wasiutyński, Jeremi. Studies in hydrodynamics and structure of stars and planets. Astrophys. Norvegica 4. Jacob Dybwad, Oslo, 1946. xvi+497 pp.

This monograph is in part a critical account of others' work, in part original research. It seeks to explain many properties of the sun, stars and planets in terms of convection, turbulence and Benard circulation. Chapter 1 is an account of turbulence, with a new evaluation of the turbulent stresses. Chapter 2 discusses criteria of instability, Eddington's circulation in stars and stellar rotation. Chapter 3 ascribes sunspots to turbulence due to nonuniform rotation in an otherwise stable layer and explains the sun's law of rotation by assuming the sun to consist of a uniformly rotating core, surrounded by an envelope much richer in hydrogen, in which the hydrodynamic circulation depends only on the distance from the equatorial plane. Chapter 4 discusses solar granulation in terms of Benard cells and chapter 5 ascribes surface features on the moon, earth and Mars to the same cause. Chapter 6 explains surface markings on Saturn and Jupiter by atmospheric belt-circulations, thermally maintained. Chapter 7 considers the structure of stars, supposed to consist of a mainly hydrogen envelope surrounding an inner part much richer in heavy elements: the possible models are classified according to the heat-transport (convective or radiative) above and below the interface between envelope and core. Giant stars are explained by a massive hydrogen envelope in radiative equilibrium surrounding a convective core of heavier material.

Of the original parts of the monograph, the main mathematical parts are (i) an incomplete discussion of currents in a rotating liquid shell in chapter 5, (ii) a discussion of belt-currents in a rotating atmosphere in chapter 6, (iii) the work of chapter 7, which depends on a large number of approximate methods. The rest deals with an enormous number of subjects, mainly descriptively. The result is stimulating, but the dangers in treating so many subjects are obvious. *T. G. Cowling* (Bangor).

Bhatnagar, P. L. Radial oscillations of a rotating star. Bull. Calcutta Math. Soc. 38, 93–95 (1946).

The motion in meridian planes is supposed wholly radial and surfaces of equal density are supposed to be similar spheroids both in equilibrium and during oscillations. Because of this over-conditioning, the period of normal oscillations is found to vary from point to point of the star. The expression for the average frequency throughout the star is not unlike that given by Ledoux [Astrophys. J. 102, 143–153 (1945); these Rev. 7, 225]. *T. G. Cowling.*

★Caldonazzo, B. Sopra alcune proprietà relative alle figure di equilibrio di liquidi rotanti. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 411–414. Edizioni Cremonense, Rome, 1942.

Barbier, Daniel. Sur la théorie du spectre continu des étoiles. Ann. Astrophysique 6, 113–135 (1943).

An approximate treatment of the equations of radiative transfer for an atmosphere in local thermodynamic equilibrium. *S. Chandrasekhar* (Williams Bay, Wis.).

RELATIVITY

Rosen, Nathan. Notes on rotation and rigid bodies in relativity theory. Physical Rev. (2) 71, 54–58 (1947).

M. Berenda a repris récemment [Physical Rev. (2) 62, 280–290 (1942); ces Rev. 4, 117] la question si la géométrie du disque tournant est euclidienne ou non et M. Hill a proposé [Physical Rev. (2) 69, 488–491 (1946); ces Rev. 8, 175] une loi non linéaire pour la vitesse des points en fonction de la distance de l'axe de rotation, parce qu'il a remarqué que la loi linéaire est en contradiction avec le principe que la vitesse de la lumière ne peut pas être dépassée. Dans ces notes M. Rosen exprime l'opinion que la difficulté envisagée par M. Hill n'est pas un argument suffisamment consistant pour abandonner la loi linéaire. Il remarque que les concepts fondamentaux de la relativité incluent l'idée de système rigide de référence et d'étalon rigide de mesure, c'est pourquoi il est désirable d'introduire des conditions covariantes pour caractériser les mouvements rigides même s'il existe un petit nombre de solutions. Le but peut être atteint en poursuivant un critère déjà posé par M. Born [Ann. Physik (4) 30, 1–56 (1909)]. L'auteur en déduit la loi linéaire de vitesse dans le problème de la rotation conformément à un résultat obtenu par M. Herglotz [Ann. Physik (4) 31, 393–415 (1910)].

Ce problème est aussi envisagé au point de vue de la distribution des vitesses angulaires en concevant le corps tournant comme un fluide, et en généralisant la relation classique $\omega=\frac{1}{2}\nabla\times v$ entre vitesse et tourbillon. On peut établir alors deux distributions de vitesses qui ne satisfont pas aux conditions covariantes de rigidité analysées précedemment; l'une d'elle avait été trouvée par M. Hill. Enfin, l'auteur complète sur un point important l'analyse de M. Berenda concernant le calcul de la distance spatiale dans un système en rotation; on en conclut que la géométrie du disque tournant n'est pas euclidienne. *G. Lampariello* (Messina).

Walker, A. G. Time-scales in relativity. Proc. Roy. Soc. Edinburgh. Sect. A. 62, 221–228 (1946).

In Milne's relativity, atomic t-time is more fundamental than gravitational τ-time, but in the present author's axiomatic development of cosmology [to appear later in the same Proc.] the τ-scale is the more fundamental. In this paper he makes a further examination of his primitive τ-scale and shows that the t-scale may be introduced by means of an axiom. Associated with t-time is the 4-dimensional model defined by $ds^2=dt^2-t^2de^2$, where de is the line-element of a 3-space of constant curvature K. World-models in which the respective values of K differ are shown to be essentially different. When a model is compared with the external world, there arise conventional τ- and t-clocks and certain conventional constants. The author discusses how these constants might be determined by actual observation. For this purpose he gives formulae for red-shift, for distance calculated from apparent size and from apparent brightness, for nebular number-counts, for sky brightness and for nebular orientation, and obtains correlations between these observables that might be used in an experimental determination of the constants. The paper is closely argued and brings out differences of standpoint between the author and Milne. *H. S. Ruse* (Leeds).

Costa de Beauregard, Olivier. Le principe de relativité et la spatialisation du temps. Rev. Questions Sci. (5) 8. 38–65 (1947).

Vallejos, M. A. Raul. Concerning the spherical space. Univ. Nac. Colombia 7, 381–396 (1946). (Spanish)

An expository discussion of relativity theory.

Schrödinger, Erwin. The general affine field laws. Proc. Roy. Irish Acad. Sect. A. 51, 41–50 (1946).

The author discusses some mathematical properties of the sets of equations obtained from the variational problem

$$\delta\int[\mathfrak{L}(R_{kl}, S_{kl})+2p^k W^{a}_{ka}]dx^1dx^2dx^3dx^4=0,$$

where $\mathfrak{L}$ is an arbitrary scalar density (unspecified throughout the paper) which is a function of the variables indicated and p^k are Lagrange multipliers. The tensor R_{kl} is the contracted curvature tensor built up from an arbitrary affine connection Δ^i_{kl} which is written as

$$\Delta^i_{kl}=\Gamma^i_{kl}+U^i_{kl},\quad \Gamma^i_{kl}=\Gamma^i_{lk},\quad U^i_{kl}=-U^i_{lk};$$

W^i_{kl} is related to U^i_{kl} through the equations

$$U^i_{kl}=W^i_{kl}+\delta^i_k V_l-\delta^i_l V_k$$

and $U^i_{ik}=3V_k$. The tensor S_{kl} is defined as

$$S_{kl}=\frac{\partial}{\partial x^k}(\Gamma^a_{al}+3V_l)-\frac{\partial}{\partial x^l}(\Gamma^a_{ak}+3V_k).$$

The variational equations obtained involve the tensor densities $\mathfrak{G}^{kl}$, $\mathfrak{F}^{kl}$ and $\mathfrak{S}^{kl}$, where

$$\mathfrak{G}^{kl}=\frac{1}{2}\left(\frac{\partial L}{\partial R_{kl}}+\frac{\partial L}{\partial R_{lk}}\right),\quad \mathfrak{F}^{kl}=\frac{1}{2}\left(\frac{\partial L}{\partial R_{kl}}-\frac{\partial L}{\partial R_{lk}}\right),\quad \mathfrak{S}^{kl}=\frac{\partial L}{\partial S_{kl}}.$$

The tensor R_{kl} is decomposed into its symmetric part γ_{kl} and its antisymmetric part φ_{kl}. Thus there are three pairs of quantities, $\mathfrak{G}^{kl}$, γ_{kl}; $\mathfrak{F}^{kl}$, φ_{kl}; $\mathfrak{S}^{kl}$, S_{kl}. The first is used to describe the gravitational field and two linear combinations of the other two pairs are used to describe the Maxwell field and the meson field. The first linear combination chosen is such that the Maxwell field has a vanishing divergence and there is no interaction between this field and the meson field. The linear combination used to describe the meson field is shown in first approximation to satisfy equations differing from Proca's equations. The author argues that this difference is due to a "slight direct influence of gravitation on the meson field and is hardly contradicted by the current view about the latter which never had occasion to contemplate such influence, direct or indirect."

A. H. Taub (Princeton, N. J.).

Einstein, A., and Straus, E. G. A generalization of the relativistic theory of gravitation. II. Ann. of Math. (2) 47, 731–741 (1946).

This is a continuation of a previous paper by Einstein [same Ann. (2) 46, 578–584 (1945); these Rev. 7, 266]. It is pointed out that the first derivation of the field equations was based on an error and a new derivation is given. The field equations of the present paper differ slightly from those first obtained.

A discussion is given of the linearized equations which hold for an antisymmetric electromagnetic field and it is shown that these equations are weaker than the corresponding Maxwell equations for empty space. The authors do not feel, however, that this provides an objection to the present theory as regular rigorous solutions which correspond to the solutions of the linearized equations are not yet known. The latter part of the paper is devoted to obtaining conditions under which the components of the linear connection are determined uniquely and without singularities. *M. Wyman* (Edmonton, Alta.).

Corben, H. C. Special relativistic field theories in five dimensions. Physical Rev. (2) 70, 947–953 (1946).

The paper deals with the mathematical formalism of a unified field theory in a flat 5-dimensional manifold whose fifth coordinate is space-like. The energy-momentum tensor and the current vector are written down in terms of a Lagrangian function. The rest-mass of a particle appears in the theory as an operator and not as an arbitrary parameter. The Lagrangians and other formal equations for particles of spin $\frac{1}{2}$, scalar particles, five-vector particles, the electromagnetic field, etc., are obtained.

G. C. McVittie (London).

Haskey, H. W. The place of Dirac's equation in five-dimensional Riemannian geometry. Proc. Edinburgh Math. Soc. (2) 7, 174–182 (1946).

Postulates are given for the formation of Dirac's equations in cylindrical 5-space of distant parallelism. Chief of these are the identification of the generalised Whittaker vector $\psi^*\gamma^\mu\psi$ with the basic vector Λ^μ of distant parallelism, the vanishing of the divergence $\Lambda^\mu{}_{;\mu}$, and the variable form of the component γ_{55} $(=\omega^2)$ of the fundamental tensor in 5-space. It is shown that $\Lambda^\mu{}_{;\mu}=0$ is equivalent to Dirac's equation and that the length of Λ^μ measures the probability of occurrence of the electron. The permissible change in the 5th coordinate is discussed and it is suggested that this is a measure of the uncertainty in the position of the electron.

A. G. Walker (Sheffield).

Snyder, Hartland S. Quantized space-time. Physical Rev. (2) 71, 38–41 (1947).

The coordinates t, x, y, z of an inertial frame are assumed to be Hermitian operators with discontinuous spectra. If η_p $(p=0, 1, 2, 3, 4)$ are real variables, then t, x, y, z are defined as the infinitesimal elements of the group which leaves the homogeneous quadratic form $\eta_0^2-\sum_{p=1}^4\eta_p^2$ invariant, so that $t=(ia/c)(\eta_4\partial/\partial\eta_0+\eta_0\partial/\partial\eta_4)$ and $ia(\eta_4\partial/\partial\eta_p-\eta_p\partial/\partial\eta_4)$ $=x, y, z$ for $p=1, 2, 3$, respectively. In these formulae, a is stated to be a natural unit of length and c to be the velocity of light, though no definition of velocity when t, x, y, z have discontinuous spectra occurs. The operators for quantum angular momentum and for energy and momentum are found. A translation of the origin of coordinates is defined as a transformation of the type $x'=\bar{S}xS$, where $\bar{S}$, S are complex conjugates. *G. C. McVittie* (London).

Dive, Pierre. Anisotropie de l'éther sur un foyer d'énergie ponctuel à symétrie sphérique en translation uniforme. C. R. Acad. Sci. Paris 223, 232–234 (1946).

By considering the principal directions of the quadratic form associated with the energy-momentum tensor the author obtains the conditions the metric tensor must satisfy in order to correspond to a spherically symmetric source of energy. *M. Wyman* (Edmonton, Alta.).

Mosharrafa, A. M. On the metric of space and the equations of motion of a charged particle. Proc. Math. Phys. Soc. Egypt 3 (1945), 19–24 (1946). (English. Arabic summary)

The author considers a Finsler 4-space with metric ds, given by

$$g_{\mu\nu}dx^\mu dx^\nu+2f_\nu dx^\nu ds-ds^2=0.$$

This metric is essentially that considered by Randers [Physical Rev. (2) 59, 195–199 (1941); these Rev. 2, 208], although the author is unaware of this earlier work. In first approximation, a geodesic of this space is identical with the world-line of a charged particle in general relativity, if $g_{\mu\nu}$ and f_μ are suitably interpreted. The short discussion on magnitudes of vectors and angles in the author's geometry is obscure and incorrect in places. *A. E. Schild.*

Narlikar, V. V., and Karmarkar, K. R. Conditions of plane orbits in classical and relativistic fields. Proc. Indian Acad. Sci., Sect. A. 24, 451–455 (1946).

General force fields are obtained which are such that the motion of a particle is plane for arbitrary initial conditions. The extension of the results to general relativity is discussed. Here the definition of "plane motion" is not invariant and by necessity somewhat arbitrary.
A. E. Schild (Pittsburgh, Pa.).

Finzi, Bruno. Formulazione integrale delle leggi meccaniche ed elettromagnetiche nello spazio-tempo. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 9(78), 204–216 (1945).

Martin, E. L. Sulla espressione del raggio nella teoria dell'universo in espansione. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 102, 455–462 (1943).

L'auteur se trouve conduit à une expansion accélerée, par l'introduction de la formule $d\delta/dl = c^{-1}R_0'/R_0 = s$ (constante positive), où δ désigne le déplacement relatif des raies, l la distance à l'origine, R_0 le rayon de l'univers à l'époque actuelle, R_0' sa dérivée par rapport au temps. Ce mode d'expansion ne semble pas devoir conduire à une interprétation physique satisfaisante. *A. Lichnerowicz.*

Martin, E. L. Il tensore contratto di curvatura per una particolare forma di cronotopo. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 102, 463–470 (1943).

Expressions des composantes du tenseur de Ricci pour la métrique $ds^2 = -\alpha dr^2 - \beta(d\theta^2 + \sin\theta d\varphi^2) + \gamma dt^2$, où α, β, γ sont des fonctions de la seule variable r. *A. Lichnerowicz.*

Martin, Ettore L. Sulle metriche relativistiche statiche a simmetria sferica. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 102, 471–482 (1943).

En utilisant des expressions obtenues dans une note précédente [voir l'analyse ci-dessus], l'auteur retrouve la solution statique la plus générale à symétrie sphérique des équations d'Einstein du cas extérieur (avec ou sans constante cosmologique). *A. Lichnerowicz* (Strasbourg).

Martin, Ettore L. Su alcuni cronotopi nella teoria dell'universo in espansione. Rend. Sem. Mat. Univ. Padova 15, 40–48 (1946).

Étude critique des théories relativistes sur l'expansion de l'univers [Lemaître, MacVittie, etc.] et notamment des formules donnant le déplacement δ des raies vers le rouge.
A. Lichnerowicz (Strasbourg).

Martin, Ettore L. Espressioni di alcune grandezze negli spazi a curvatura costante. Rend. Sem. Mat. Univ. Padova 15, 49–59 (1946).

Recueil de formules donnant les expressions des principaux éléments géométriques pour un espace tridimensionnel à courbure constante positive, nulle ou négative.
A. Lichnerowicz (Strasbourg).

MECHANICS

***Newboult, H. O. Analytical Method in Dynamics.** Oxford University Press, London, 1946. vii+81 pp. $2.25, U. S. A.; 7/6, Great Britain.

As stated in the preface, this expository book is intended to supplement rather than to replace standard works on dynamics. Five of the six chapters are devoted to the discussion of moving frames of reference and to applications to problems in the dynamics of particles and, more especially, of rigid bodies. Curiously enough, sandwiched between chapter IV on the motion of the top and chapter VI on the motion of a rigid body about a fixed point under no forces, appears the one exceptional chapter V on the unrelated subject of the motion of an inextensible flexible chain. The word, "true," as applied to the rate of change of a vector whose components are given with respect to moving axes, is first used on page 6 and in a more general connection on page 26. In view of the fundamental importance of this concept, it is unfortunate that no formal definition is given of its meaning. Aside from this defect, the treatment of angular velocity as a vector quantity and the discussion of the familiar vector equation $dr/dt = \dot{r} + \omega \times r$ is exemplary, though perhaps longer than necessary. There follows a very good treatment of Euler's angles and of the three-dimensional kinematics and kinetics of a rigid body. Later Lagrange's equations are used but are nowhere proved. Although a number of the problems involve such complications as nonholonomic constraints, the book contains no mention of general methods for eliminating the forces of constraint, nonholonomic or otherwise. There are about a dozen interesting problems whose solutions are worked out in detail. *D. C. Lewis* (College Park, Md.).

***Chazy, Jean. Cours de Mécanique Rationelle. Tome I. Dynamique du Point Matériel.** 3d ed. Gauthier-Villars, Paris, 1947. v+482 pp.

This text on the dynamics of a material particle begins with a discussion of vectors. The principles of mechanics are then formulated and in successive chapters applied to problems of equilibrium, motion in one, two and three dimensions, and later to motion on a curve and in a surface. The effects of friction and the rotation of the earth are discussed.

The treatment is clear and logical with special attention devoted to qualitative results and the precise consideration of singular situations where the simplest existence theorems do not apply. There is an underemphasis on specific illustrative problems worked out in full detail and results obtained by numerical calculation. There are no exercises throughout the book, but at the end the author has included the examinations in his subject for the years 1928–1946.
P. Franklin (Cambridge, Mass.).

***Sestini, Giorgio. Composizioni di moti rigidi.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 382–388. Edizioni Cremonense, Rome, 1942.

✱Arrighi, Gino. Le soluzioni statiche delle equazioni dei piccoli movimenti attorno una configurazione di equilibrio stabile di tipo generale. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 389–392. Edizioni Cremonense, Rome, 1942.

✱Venturelli, Lucia. Sul moto di un sistema rigido pesante un punto del quale è vincolato a una retta fissa. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 393–403. Edizioni Cremonense, Rome, 1942.

Mercalov, N. I. The problem of the motion of a solid body, having a fixed point, for $A=B=4C$ and the area integral $\neq 0$. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Acad. Nauk SSSR] **1946**, 697–701 (1946). (Russian)

L'auteur étudie le problème classique du mouvement d'un solide S, soumis à l'action de la pesanteur seule et dont le point O est fixe. Il construit une solution approchée des équations d'Euler, en particularisant l'ellipsoïde d'inertie Σ de S en O ainsi que les conditions initiales. Avec les notations classiques, il admet que: (I) $A=B=4C$, (2) le centre de gravité de S est dans le plan équatorial de Σ, (3) $p_0=q_0=0$. La constante des aires n'est pas supposée nulle. L'auteur espère réaliser un gyroscope propre à servir au contrôle expérimental des lois de mouvement qu'il a trouvées.

J. Kravtchenko (Grenoble).

Thiry, René. Sur le sens de la vitesse moyenne de précession du gyroscope. C. R. Acad. Sci. Paris **223**, 1081–1082 (1946).

Dynamical (as opposed to the usual more or less purely analytical) considerations are produced to prove the well-known theorem that the mean value of the precession velocity has the same sign as its instantaneous value when the nutation angle is a maximum. *D. C. Lewis.*

✱García, Godofredo. Sobre el Problema Balístico del Proyectil-Cohete. [On the Ballistic Problem of the Rocket]. Imp. de la Escuela Militar, Chorrillos, 1947. 20 pp.

An exposition applying methods initiated by Poincaré and continued by Picard and Popoff for obtaining series expansions for the ballistic trajectory in terms of the sine of the angle of departure, and time. The paper touches upon the major mathematical problems of exterior ballistics as affecting rockets. Expansions are carried formally to second degree terms. No numerical results are shown.

A. A. Bennett (Providence, R. I.).

Costa de Beauregard, Olivier. Sur les théorèmes généraux de la dynamique. C. R. Acad. Sci. Paris **224**, 540–541 (1947).

The author treats systems of fourth order vectors and skew-symmetric tensors, with applications to the motion of mass points in a field. *P. Franklin* (Cambridge, Mass.).

Mattioli, Gian Domenico, and Carrelli, Antonio. Sul significato del principio di Hamilton. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 142–157 (1940).

The authors consider the question of whether the stationary value of Hamilton's integral is a minimum value and for this purpose discuss an example, given by Born [Proc. Roy. Inst. **30**, 1596–1628 (1939)], on the bending of an elastic lamina, leading to the well-known result that $\int_{t_1}^{t_2}$ has a minimum only if t_2 is less than a certain finite value. [A more elementary example can be given by considering the motion of a particle under no forces on a smooth spherical surface. On pp. 144–146, starting from three "principles," the authors derive the equations of motion by a deduction which is not convincing to the reviewer; cf., in particular, equation (5), p. 145.] *O. Bottema* (Delft).

Lichnerowicz, André. Les relations intégrales d'invariance et leurs applications à la dynamique. Bull. Sci. Math. (2) **70**, 82–95 (1946).

Given a system of differential equations

$$dx_1/X_1=\cdots=dx_m/X_m,$$

where the X's are functions of the x's, "an integral relation of invariance" is said to be generated by an exterior differential form ω of degree $p+1$ whenever $\int\omega=0$ for an arbitrary $(p+1)$-dimensional region made up of a p-parameter family of trajectory arcs of the differential equations. It is obvious from the generalized formula of Stokes that ω generates an integral relation of invariance if, and only if, it is the first derived form of a relatively invariant integral. The paper consists largely of generalizations and detailed proofs of a previous note by the same author [C. R. Acad. Sci. Paris **217**, 660–662 (1943); these Rev. **6**, 243]. *D. C. Lewis.*

Mineur, Henri. Sur les systèmes mécaniques dont les intégrales premières sont définies par des équations implicites. C. R. Acad. Sci. Paris **224**, 26–27 (1947).

It is pointed out that the Hamiltonian system

$$dq_i/dt=\partial H/\partial p_i, \quad dp_i/dt=-\partial H/\partial q_i, \qquad i=1,\cdots,n,$$

where H is defined implicitly by

$$F(H,t,q_1,\cdots,q_n,p_1,\cdots,p_n)=0,$$

is parametrized by those solutions of the system

$$dH/d\tau=\partial F/\partial t, \quad dt/d\tau=-\partial F/\partial H,$$
$$dq_i/d\tau=\partial F/\partial p_i, \quad dp_i/d\tau=-\partial F/\partial q_i, \qquad i=1,\cdots,n,$$

for which the invariant function F is zero. This idea is generalized to the case when there are k integrals in involution defined implicitly by means of k equations.

D. C. Lewis (College Park, Md.).

Hydrodynamics, Aerodynamics, Acoustics

Possio, Camillo. Campo di velocità creato da un vortice in un fluido pesante e superficie libera in moto uniforme. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **76**, 365–388 (1941). [MF 16262]

The flow field due to a horizontal line vortex with circulation Γ, combined with a uniform velocity V_0 perpendicular to the vortex line, is investigated in an infinite inviscid incompressible liquid with a free horizontal surface. Using the method of images the author finds that the potential Φ is given by

$$\Phi=V_0x+\frac{\Gamma}{2\pi}\left\{\tan^{-1}\frac{y-h}{x}+\tan^{-1}\frac{x}{y+h}\right\}-\Gamma e^{-\beta(1+y/h)}\cos\frac{\beta x}{h}$$
$$+\frac{\Gamma}{\pi}\int_0^\infty e^{-m\beta(1+y/h)}\frac{\sin m\beta x/h}{1-h}dm$$

(h depth of vortex line below water surface, $\beta=gh/V_0^2$, y depth measured from water surface, x horizontal coordinate measured from vortex line). The velocity field is evaluated. A study of the case of a vortex distribution $\gamma(\xi,\eta)$ is added. *P. Neményi* (Washington, D. C.).

★Finzi, Bruno. Un teorema di minimo nella meccanica dei liquidi viscosi. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 415–421. Edizioni Cremonense, Rome, 1942.

Meksyn, D. Stability of viscous flow between rotating cylinders. I. Proc. Roy. Soc. London. Ser. A. **187**, 115–128 (1946).

The author studies the stability of viscous flow between rotating cylinders by using asymptotic solutions. Following G. I. Taylor, the disturbance is assumed to be proportional to $e^{\sigma t}\cos\lambda z$, where λ and σ are constants, t is the time and z is the distance along the axis of symmetry of the cylinder. The results confirm those obtained by Taylor (both experimentally and theoretically). In addition, the author shows that, for a fixed value of the angular velocity of the inner cylinder, there is only a limited number of possible wavelengths $2\pi/\lambda$. It is also shown that there are no solutions for complex or imaginary σ. The discussion in this paper is limited to the case when the cylinders rotate in the same direction. The more general case is considered in the paper reviewed below. *C. C. Lin* (Providence, R. I.).

Meksyn, D. Stability of viscous flow between rotating cylinders. II. Cylinders rotating in opposite directions. Proc. Roy. Soc. London. Ser. A. **187**, 480–491 (1946).

The author extends the study begun in the paper reviewed above to include the case where the cylinders are rotating in opposite directions. The asymptotic solutions exhibit Stokes's phenomenon in the range of interest. By studying the proper transformations [see part III, same vol., 492–504 (1946); these Rev. **8**, 382] the author is able to reach the following conclusions in agreement with Taylor's experimental work. The spacing of the vortices and the critical angular speed Ω_1 of the inner cylinder satisfy the equations $\lambda d_1=\pi/1.546$ and

$$4.053/d_1=8\Omega_1^2(1-\mu R_2^2/R_1^2)^2/\nu^2\lambda^4(R_2^2/R_1^2-1)^2R_0,$$

where $d_1=R_0-R_1$; R_1 and R_2 $(R_2>R_1)$ are the radii of the cylinders, R_0 is the radius where the mean velocity vanishes and $\mu=\Omega_2/\Omega_1$, the ratio of the angular velocities of the cylinders; $2\pi/\lambda$ is the wave-length parallel to the axis of the cylinder; ν is the kinematic viscosity coefficient. *C. C. Lin* (Providence, R. I.).

Carrier, G. F. The boundary layer in a corner. Quart. Appl. Math. **4**, 367–370 (1947).

The author investigates the flow of a viscous fluid in the corner between two semi-infinite planes at large Reynolds numbers. The problem is thus a generalization of the Blasius solution for the boundary layer along a flat plate. Let u be the velocity in the direction of the mean flow, v and w the components normal to the mean flow; $\eta=y(v/\nu x)^{\frac{1}{2}}$, $\xi=z(v/\nu x)^{\frac{1}{2}}$ are the proper dimensionless variables. The problem leads to the equation $g_{\eta\eta\eta\xi}+g_{\eta\xi\xi\xi}+\frac{1}{2}(g_\xi g_{\eta\eta\xi}+g_\eta g_{\eta\xi\xi})=0$, where $u=g_{\eta\xi}$, $v=\frac{1}{2}(\eta g_{\eta\xi}-g_\xi)$, $w=\frac{1}{2}(\xi g_{\eta\xi}-g_\eta)$. The solution must approach Blasius' case for large η,ξ and hence it is convenient to write $g(\eta,\xi)=f(\eta)f(\xi)+h(\eta,\xi)$, where f is Blasius' function and satisfies $2f'''+ff''=0$. The numerical solution is then obtained by means of the relaxation method. *H. W. Liepmann* (Pasadena, Calif.).

Busemann, Adolf. Infinitesimal conical supersonic flow. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1100**, 10 pp. (6 plates) (1947).

Translation of the paper reviewed below.

Busemann, Adolf. Infinitesimale kegelige Überschallströmung. Schr. Deutsch. Akad. Luftfahrtforschung **7B**, 105–121 (1943).

This paper considers the conical fields of flow obtained from the linearized potential equation of supersonic gas flow. In conical coordinates, $\xi=x/z$, $\eta=y/z$, where z is the direction of the basic uniform flow, the differential equation for the potential of the conical flow is linear and of elliptic type within the Mach cone defined by $\xi^2+\eta^2=\tan^2\alpha$, α the Mach angle. A suitable transformation of coordinates [which is essentially that used by Chaplygin for this purpose] reduces the equation to Laplace's equation and the flow problem therefore to one of potential theory. This method is applied to the solution of several flow problems, including the flow past a cone, rectangular plate, and lifting triangle. The possibility of superposition of conical fields with different origins is shown to increase the applicability of the method. *D. Gilbarg* (Bloomington, Ind.).

Gurevich, M. I. Lift force of an arrow-shaped wing. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 513–520 (1946). (Russian. English summary)

If a compressible flow has the velocity vector $(u, v, W+w)$, W=constant, and the ratios u/W, v/W, w/W are assumed to be small of first order, then u, v, w satisfy the Prandtl-Glauert equation $\varphi_{xx}+\varphi_{yy}+(1-M^2)\varphi_{zz}=0$, M being the stream Mach number and $(u, v, w)=\operatorname{grad}\varphi$. If $M>1$ and the flow is conical (i.e., u, v, w are functions of $\xi=x/z$, $\eta=y/z$), then the determination of the flow within the Mach cone $\xi^2+\eta^2<A^2$, $A^2=1/(M^2-1)$, reduces to the integration of the Laplace equation.

Set $\xi+i\eta=Re^{i\varphi}$, $R=2A\epsilon/(1+\epsilon^2)$, $\tau=\epsilon e^{i\varphi}$. Then w is the real part of an analytic function $w+is=Af(\tau)$, $|\tau|\leqq 1$, and $u+iv=-\frac{1}{2}\int\{df+\tau^{-1}df\}$. The author derives these relations and establishes the boundary conditions for w and s in the case of a flow past a plane slightly inclined delta-wing, contained or not within the Mach cone. The corresponding functions $w+is$ can be constructed by standard methods and explicit expressions are obtained for the lift coefficient C_y. If the wing is not contained within the Mach cone, $C_y=4\beta(M^2-1)^{-1}$, β being the angle of attack [cf. Ackeret, Z. Flugtech. Motorluftschiffahrt **16**, 72–74 (1925)]. The expression for a wing contained within the Mach cone is more complicated and involves elliptic integrals. *L. Bers* (Syracuse, N. Y.).

Germain, Paul. Étude de certains régimes coniques. C. R. Acad. Sci. Paris **224**, 183–185 (1947).

The linearized equation of supersonic flow is transformed, for the special case of "conical flow," into the Laplace equation in two dimensions. The relations for irrotational flow give a set of differential relationships between the functions whose real parts are the Cartesian velocity components. This treatment is analogous to that of Stewart [Quart. Appl. Math. **4**, 246–254 (1946); these Rev. **8**, 109]. The case of the triangular wing ("delta wing") of wedge-shaped profile, at zero incidence, is treated. The result, in the form of the velocity at the wing surface, is the same as that of Puckett [J. Aeronaut. Sci. **13**, 475–484 (1946); these Rev. **8**, 109]. *W. R. Sears* (Ithaca, N.Y.).

Frankl, F. I. Uniqueness of solution of the problem of supersonic flow past a wedge. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 421–424 (1946). (Russian. English summary)

This is a sequel to a previous paper [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 121–

143 (1945); these Rev. 7, 496] in which the author formulated boundary value problems for Chaplygin's hodograph equation leading to two types of trans-sonic two-dimensional gas-flows: (a) trans-sonic flow from a vessel bounded by two straight walls, (b) trans-sonic flow past a wedge with a shock-wave not touching the wedge. In this note the corresponding boundary value problems are stated for the case of a flow past the wedge with the shock-wave starting at the tip of the wedge. The nonuniformity of entropy changes across the shock is neglected and the flow behind the shock is assumed to be irrotational. The boundary value problems are [as in the preceding paper] either of the Tricomi type or a generalization of the Tricomi problem. Using geometrical considerations concerning the mapping from the hodograph plane to the physical plane the author shows that his boundary value problems in the hodograph plane are the only ones yielding the desired flows in the physical plane. This explains why in the wedge problem only one of the two intersections of the line $\theta=$constant with the shock-polar has physical meaning [cf. Levinson, same journal 9, 151–170 (1945); these Rev. 8, 111] and why, in the case of flow (a), the line $M=1$ (M, the Mach number) passes through the points of intersection of the fixed and free boundaries. *L. Bers* (Syracuse, N. Y.).

Frankl, F. I. Theory of a propeller with a finite number of blades at high speeds of advance and revolution. C. R. (Doklady) Acad. Sci. URSS (N.S.) 53, 405–408 (1946).

This is a brief summary of a "first order" theory of an airscrew with a finite number of blades which takes compressibility into account. The flight speed is subsonic and the velocity disturbances are assumed to be small in comparison to the speed of the blade tips. Vortex sheets along helical surfaces described by the blades are assumed and a coordinate system fixed with respect to the air at infinity is used. The disturbance potential is given in the form of a retarded potential of a double layer spread over the vortex-sheets. It is a (nonunique) solution of the wave-equation satisfying the following boundary conditions: (1) the disturbances vanish far ahead of the propeller, (2) the jumps in the potential are equal to the given circulation, (3) the pressure is continuous across the vortex-sheets, (4) the flux through any closed surface rigidly connected with the propeller vanishes. Reference is made to an earlier paper by the author unavailable to the reviewer [Trans. Central Aero-Hydrodynamical Inst., no. 540 (1941)]. *L. Bers.*

Frankl, F. To the theory of the Laval nozzle. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 387–422 (1945). (Russian. English summary)

The author describes a method of computing two-dimensional flows in Laval nozzles with transition from subsonic to supersonic speeds and without shock-waves. The flows are assumed to be symmetrical with respect to the x-axis. The intersection of the line $M=1$ (critical line; M, Mach angle) and the x-axis is called the center of the nozzle and is taken as the origin of the (x, y) coordinate system. At first the author assumes that the stream-function ψ of the flow can be expanded in a power series in x and y in the neighborhood of the center. From this expansion he concludes that ψ, considered as a function of the speed w and the velocity-slope θ, satisfies (for $M=1$) the conditions $\psi=3^{\frac{1}{3}}(\gamma+1)^{\frac{1}{3}}\alpha^{-1}+O(\theta)$, $\psi/w=-3^{-\frac{1}{3}}(\gamma+1)^{-\frac{1}{3}}\alpha^{-1}\theta^{-\frac{1}{3}}+O(\theta)$, as well as similar conditions for $\partial\psi/\partial\theta$ and $\partial^2\psi/\partial w\partial\theta$. These conditions are called "continuation conditions." Here γ is the ratio of specific heats and α the x-derivative of w at the center.

Let a function ψ be given in a part of the subsonic region of the hodograph plane adjacent to the line $M=1$ and let ψ satisfy Chaplygin's equation and yield in the physical plane a flow which is symmetrical with respect to the streamline $y=0$. The author shows that this function can be continued into the supersonic region in such a way as to yield a flow of the desired type in the physical plane, provided that ψ satisfies the "continuation conditions." The continuation of ψ is not uniquely determined. In fact, it is possible to prescribe the values of $u(x, 0)$, $x>0$, u being the x-component of the velocity. It is permitted to require u_x to be discontinuous at $x=y=0$. However the values $\alpha_1=u_x(-0, 0)$, $\alpha_2=u_x(+0, 0)$ must satisfy the inequalities (*) $\alpha_2\leqq\alpha_1\leqq4\alpha_2$.

The proof is rather complicated. Among other things it requires a careful discussion of the connection between the physical and hodograph planes in the supersonic part of the neighborhood of the center of the nozzle. There ψ is a partly three-valued function of the hodograph variables and the hodograph-image of the physical plane is a partly triply covered surface possessing "branch lines" along characteristics. An essential tool of the proof is the theory of the Cauchy problem for the Chaplygin equation given by the author in a previous paper [same Bull. 8, 195–224 (1944); these Rev. 7, 17]. In the supersonic region the function ψ can be represented in the form $\psi=\psi_{\mathrm{I}}+\psi_{\mathrm{II}}$, where ψ_{I} and its derivatives are considerably larger than ψ_{II} and its derivatives for small values of $M-1$. The "principal term" ψ_{I} is a solution of the Darboux-Tricomi equation with appropriate boundary conditions. It is given by explicit formulas in terms of hypergeometric functions.

Finally the author exhibits a three-parameter family of functions ψ defined in the subsonic region of the hodograph plane and satisfying the "continuation conditions." These functions are given by the formula

$$\psi=-\theta-A\sum_{n=1}^{\infty}\{z_{cn}(\tau)/z_{cn}(\tau^*)\}n^{-4/3}\sin 2cn\theta,$$

where the notations used are those of Chaplygin [Učenye Zapiski Imp. Moskov. Univ., Otd. Fiz.-Mat. 21, 1–121 (1904); translation in Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1063 (1944); these Rev. 7, 495]. To verify the "continuation conditions" the author investigates series of the form $\sum n^{-2k/3}\sin n\theta$ and derives an asymptotic formula for Chaplygin's function $z_\nu(\tau^*)$ for large values of ν. [Some corrections and simplifications are indicated in a more recent paper by the author, same Bull. 10, 135–166 (1946); these Rev. 8, 77.] *L. Bers* (Syracuse, N. Y.).

Falkovich, S. V. On the theory of the Laval nozzle. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 10, 503–512 (1946). (Russian. English summary)

This paper deals with the same problem as Frankl's paper reviewed above. While Frankl works with the stream-function ψ considered as a function of conveniently chosen hodograph variables and obtains a representation of ψ in the neighborhood of the center of the nozzle as a sum of a "principal term" satisfying Tricomi's equation (and given by hypergeometric functions) and a small correction term, Falkovich considers a certain function $\eta(w)$ of the speed w and the velocity-slope θ as functions of ψ and the velocity potential ϕ. The equations of motion take the form

(1) $\partial\eta/\partial\psi+b(\eta)\partial\theta/\partial\phi=0,\quad \eta\partial\eta/\partial\phi-\{b(\eta)\}^{-1}\partial\theta/\partial\psi=0,$

where $b(\eta)$ is a given function. For $M=1$ (M, Mach number), $\eta=0$, and for small values of η equations (1) are replaced by the linear system

$$(2) \qquad \partial\eta/\partial\bar{\psi}+\partial\theta/\partial\phi=0, \quad \eta\partial\eta/\partial\phi-\partial\theta/\partial\bar{\psi}=0,$$

where $\bar{\psi}=b(0)\psi$. A particular solution is given by

$$(3) \qquad \theta=A^2\phi\bar{\psi}-\tfrac{1}{3}A^3\bar{\psi}^3, \quad \eta=A\phi-\tfrac{1}{2}A^2\bar{\psi}^2.$$

It turns out that this is the "principal term" of the nozzle-flow solution of system (1), the constant A being proportional to the value of $\partial w/\partial x$ at the center. The simplicity of formulas (3) permits a comparatively easy determination of stream-lines and Mach lines near the nozzle center, as well as an analysis of the peculiar connection between the physical and hodograph planes. If a "weak discontinuity" is admitted, A has two different values for $M<1$ and $M>1$ and the Frankl condition for the possibility of a shockless flow [inequalities (*) of the preceding review] is derived in a simple manner. *L. Bers* (Syracuse, N. Y.).

Puckett, A. E. Supersonic nozzle design. J. Appl. Mech. 13, A-265–A-270 (1946).

The aim of this article is to present the characteristics method for computing (two-dimensional) supersonic gas-flows in a manner appealing to the engineer and leading immediately to practical application. No use is made of partial differential equations. Instead it is stated that a supersonic flow field may be approximated by a field consisting of small quadrilaterals of uniform flow. The transition from one quadrilateral to the other (i.e., the change of flow across a disturbance wave) can be computed from the basic conservation laws and is uniquely determined by the change in flow-direction. Interaction of waves and walls is considered and the design of supersonic nozzles producing uniform flow at the end is treated in detail. A brief section is devoted to boundary layer corrections. *L. Bers.*

Tsien, Hsue-shen. One-dimensional flows of a gas characterized by van der Waal's equation of state. J. Math. Phys. Mass. Inst. Tech. 25, 301–324 (1947).

The one-dimensional steady flow of a gas in a nozzle is investigated under van der Waal's equation of state $p=RT(\rho^{-1}-b)^{-1}-a\rho^2$. A quadratic expression

$$C_V=\alpha+\beta T+\gamma T^2$$

for the absolute temperature T is assumed for the specific heat C_V at constant volume. Using the fundamental equations $C_VdT+pd\rho^{-1}=0$, $\rho vdv+dp=0$, and $\rho vA=\text{constant}$, where A denotes the cross-sectional area of the nozzle, the nondimensional quantities T/T_0, ρ/ρ_0, p/p_0, A/A^*, where A^* is the cross-sectional area at the throat, are calculated in terms of the Mach number M under the assumption that the squares and higher powers of a, b, β, γ can be neglected. Under the same assumption, the ratios p_2/p_1, ρ_2/ρ_1, T_2/T_1 and the Mach number M_2 after a normal shock are expressed in terms of the initial conditions of the shock and the parameters α, β, γ, a, b. The theory is illustrated by numerical examples and tables of correction factors for air are included. The author states that "even for a Mach number as low as 3, the deviations from the perfect gas law cannot be neglected if the correction parameters are not extremely small." *M. H. Martin* (College Park, Md.).

Tsien, Hsue-shen. Corrections on the paper "One dimensional flows of a gas characterized by Van der Waal's equation of state." J. Math. Phys. Mass. Inst. Tech. 26, 76–77 (1947).

The author lists corrections to the paper reviewed above; the variation of internal energy of a gas caused by change in volume was overlooked there. *M. H. Martin.*

Sretenskiĭ, L. N. The flow of gas jets around a plane contour. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] 1945, 622–637 (1945). (Russian)

According to Chaplygin, the streamfunction ψ and the potential function ϕ of two-dimensional steady flow of a compressible fluid are connected by the relations

$$\varphi_\theta=(1-\lambda\tau)^{-\beta}\psi_\omega, \quad \varphi_\omega=-\{1-(2\beta+1)\lambda\tau\}(1-\lambda\tau)^{-(1+\beta)}\psi_\theta.$$

Here β is a gas constant; $\tau=\exp(-2\omega)$ a certain function of the speed v; θ the angle which the velocity vector forms with a fixed direction, say with the positive x-axis; $\lambda=B^2(2\beta+B^2)^{-1}$, where B is the Mach number at infinity. For subsonic flows λ varies in the range between 0 and $\frac{1}{2}$. The author represents the solution in the form $\varphi=\sum_{n=0}^{\infty}\lambda^n\varphi_n$, $\psi=\sum_{n=0}^{\infty}\lambda^n\psi_n$ and obtains successive equations for the determination of the φ_n's and ψ_n's. Substituting for $\varphi_0+i\psi_0$ the complex potential of an incompressible fluid and replacing the infinite sum by a finite one, he obtains an approximate method for the determination of flow patterns. As an example for the method, a flow around a circular cylinder is determined.

The reviewer would like to mention that the operation $(\varphi, \psi)=\mathrm{P}(\varphi_0, \psi_0)$ given by the author for transforming the complex potential $\varphi_0+i\psi_0$ of an incompressible fluid into that of a compressible fluid has a similarity with the operator indicated by the reviewer [Tech. Notes Nat. Adv. Comm. Aeronaut., nos. **972** (1945), **1018** (1946); these Rev. **7**, 342; **8**, 295]. *S. Bergman* (Cambridge, Mass.).

Nughin, S. G. Theory of flow of gas past a body at high subsonic velocities. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 657–666 (1946). (Russian. English summary)

The author proposes an approximate pressure correction formula for subsonic flows: $P^*=f(P, M)$, P^* being the local pressure in a flow past a body with stream Mach number M, P the pressure at the same point in an incompressible flow ($M=0$). The function $f(P, M)$ is tabulated for $M=.1$, $.15, \cdots, .85$. Examples of pressure distributions are given and excellent agreement with experimental results is claimed. The heuristic idea underlying the author's procedure consists in interpreting the mass-flow vector of a potential gas-flow as the velocity vector of a rotational flow of an incompressible fluid. *L. Bers* (Syracuse, N. Y.).

Gelbart, Abe. On subsonic compressible flows by a method of correspondence. I. Methods for obtaining subsonic circulatory compressible flows about two-dimensional bodies. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1170**, 35 pp. (1947).

The first part of this paper [submitted to the NACA in July, 1945] deals with two-dimensional potential adiabatic flows (with the pressure density relation $p\rho^{-\gamma}=\text{constant}$). Such flows admit the Chaplygin representation

$$(1) \qquad z=\int q^{-1}e^{i\theta}(d\varphi+i\rho^{-1}d\psi),$$

where $z=x+iy$ is the complex variable in the physical

plane, $qe^{i\theta}$ the complex velocity, $\rho=\rho(q)$ the density and $\varphi+i\psi$ the complex potential. In the hodograph plane φ and ψ are connected by equations of the form $\varphi_\theta=\tau_1(q)\psi_q$, $\varphi_q=-\tau_2(q)\psi_\theta$; thus $\varphi+i\psi$ is a Σ-monogenic function [Bers and Gelbart, Quart. Appl. Math. 1, 168–188 (1943); Trans. Amer. Math. Soc. 56, 67–93 (1944); these Rev. 5, 25; 6, 86]. The author describes a "correspondence method" for the choice of a Σ-monogenic function yielding a flow with desired properties. It is mentioned that many known methods for generating compressible flow patterns are essentially identical with this correspondence method. The difficulties of the method are discussed and two specific analytic representations of Σ-monogenic functions are used to obtain expressions for z in (1): the Σ-Taylor series [see the first reference above] and the Σ-Laplace transforms to be discussed in a forthcoming paper by the author and the reviewer.

In the second part of the paper the idea of the correspondence method is applied to flows obeying the linearized pressure-density relation ($\gamma=-1$). The author obtains the general parametric representation

$$z=f(\zeta)-\tfrac{1}{4}\overline{\int[G'(\zeta)^2/f'(\zeta)]d\zeta}, \tag{2}$$

$$q=|G'/f'|\{1-\tfrac{1}{4}|G'/f'|^2\}^{-1}. \tag{3}$$

Here ζ is an auxiliary complex variable, f an arbitrary analytic function, G the complex potential of an incompressible flow in the ζ-plane. If a flow past a closed body is desired G may be taken as the potential of a flow (in general circulatory) past a circle and $f(\zeta)$ is to have the form

$$f(\zeta)=b_{-1}+b_0\zeta+b_1\log\zeta+b_1\zeta^{-1}+b_2\zeta^{-2}+\cdots. \tag{4}$$

Here b_0 is determined by the conditions at infinity, b_1 by the condition that the profile in the z-plane is closed. The remaining coefficients determine the shape of the body. Formula (2) contains as a special case the formulas previously given by Tsien for flows without circulation [J. Aeronaut. Sci. 6, 399–407 (1939); these Rev. 2, 168] and by the reviewer for flows with circulation [Tech. Notes Nat. Adv. Comm. Aeronaut., no. **969** (1945); these Rev. 7, 497].

L. Bers (Syracuse, N. Y.).

Bartnoff, Shepard, and Gelbart, Abe. On subsonic compressible flows by a method of correspondence. II. Application of methods to studies of flow with circulation about a circular cylinder. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1171**, 34 pp. (5 plates) (1947).

The method of the paper reviewed above is applied to the flow past a circular cylinder. The linearized pressure-volume relation is assumed. It is shown how the coefficients in (4) must be chosen in order to obtain shapes closely approximating a circle. *L. Bers* (Syracuse, N. Y.).

Lin, C. C. On an extension of the von Kármán-Tsien method to two-dimensional subsonic flows with circulation around closed profiles. Quart. Appl. Math. 4, 291–297 (1946).

The author derives a parametric representation of a potential compressible flow past a closed body, assuming that the pressure of the fluid is a linear function of the specific volume ($\gamma=-1$). The representation is identical with that given by Gelbart [see the second preceding review; Lin's paper was received by the editors in May, 1946]. The representation permits construction of circulatory flows and extension to compressible flows of von Mises' method of generating profiles. Further applications are to be discussed in a subsequent paper. The author indicates that the direct problem (computation of a flow past a given body) is equivalent to a mapping problem which he hopes might be attacked by the method of successive approximations. [The reduction of the direct problem to a mapping problem and to an integral equation which can be solved by successive approximations was given earlier in a paper by the reviewer dealing with circulation-free flows, Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1006** (1946); these Rev. 8, 107.]

L. Bers (Syracuse, N. Y.).

Kalikhman, L. E. Heat transmission in the boundary layer. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 10, 449–474 (1946). (Russian. English summary)

[A more accurate translation of the Russian title is "The gas dynamic theory of heat transfer."] There is developed a general analytical procedure for extending present methods of boundary layer analysis to the case of compressible flow. Both laminar and turbulent flows are considered. The essential device in this extension is the introduction of a new independent variable $\eta=\int_0^y(\rho/\rho_0)dy$ for the coordinate normal to the wall, where ρ/ρ_0 is the ratio of density in the boundary layer to the corresponding isentropic stagnation density. In obtaining this density ratio the effect of the heat transfer through the wall is included in the expression for the temperature even though the magnitude of this effect is dependent upon the ultimate determination of the boundary layer characteristics. In the case of laminar boundary layers the general equations are transformed using this new variable and, in particular, the method of Polhausen is extended to obtain specific solutions. The same change of variable is applied to turbulent boundary layers, extending the well-known concept of turbulent exchange coefficient to the analysis of friction and heat transfer.

The development in this paper is very thorough starting from established fundamentals. The working procedures should be applicable to a variety of problems involving compressible boundary layers with heat transfer.

N. A. Hall (East Hartford, Conn.).

Dryden, Hugh L. Some recent contributions to the study of transition and turbulent boundary layers. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1168**, 32 pp. (12 plates) (1947).

Expository lecture.

Ivey, H. Reese, and Bowen, Edward N., Jr. Theoretical supersonic lift and drag characteristics of symmetrical wedge-shape-airfoil sections as affected by sweepback outside the Mach cone. Tech. Notes Nat. Adv. Comm. Aeronaut. no. **1226**, 16 pp. (44 plates) (1947).

Bonney, E. Arthur. Aerodynamic characteristics of rectangular wings at supersonic speeds. J. Aeronaut. Sci. 14, 110–116 (1947).

Omara, M. A. Hydrodynamic forces on a cylinder moving in a compressible fluid. Proc. Math. Phys. Soc. Egypt 3 (1945), 53–63 (1946). (English. Arabic summary)

This is an attempt to calculate the force and moment acting on a cylinder of arbitrary profile in general uniplanar motion in a compressible fluid. Certain steps in the derivation seem to limit the theory to motion at subsonic speeds and this is confirmed by the form of the results; they are analogous to the corresponding formulas for incompressible fluids. *W. R. Sears* (Ithaca, N. Y.).

Schade, Th., and Krienes, K. The oscillating circular airfoil on the basis of potential theory. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1098, 34 pp. (4 plates) (1947).

Translation of articles in Luftfahrtforschung **17**, 387–400 (1940) [Schade]; **19**, 282–291 (1942) [Krienes]; these Rev. **3**, 285; **4**, 177.

Watkins, Charles E. The streamline pattern in the vicinity of an oblique airfoil. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1231, 7 pp. (19 plates) (1947).

Udeschini, Paolo. Incompatibilità dell'adesione completa al contorno con la regolarità e le condizioni asintotiche euleriane per correnti viscose stazionarie. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 957–963 (1941).

Consider a solid S immersed in a steady Eulerian current of viscous fluid (compressible or incompressible) of asymptotic velocity $\mathbf{c}$. Make only the following hypotheses: (1) the motion is regular, i.e., there are no singularities; (2) $\lim_{r\to\infty} r^2(\mathbf{v}-\mathbf{c})=0$, where $\mathbf{v}$ is the velocity vector. The author then gives a proof that the supposition $\mathbf{v}=0$ at the surface of S, i.e., complete adhesion of the fluid, is incompatible with (1) and (2). The proof assumes conservative body force. The result is not surprising, but the proof requires considerable amplification to justify the inversion of limiting processes and the differentiation of infinite energies. *L. M. Milne-Thomson* (Greenwich).

Stepanov, E. I. Sur la rotation lente des solides dans liquide visqueux. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 673–675 (1946). (Russian. French summary)

Cet article est consacré à l'analogie entre le problème mentionné dans le titre et celui du mouvement de translation d'un solide de révolution le long de son axe de symétrie dans un liquide parfait. *Author's summary.*

Levinson, Norman. On the asymptotic shape of the cavity behind an axially symmetric nose moving through an ideal fluid. I. Ann. of Math. (2) **47**, 704–730 (1946).

The author considers the asymptotic shape of the cavity formed behind the nose of an axially symmetric solid body moving with uniform velocity in a direction parallel with its axis through a homogeneous perfect fluid. The surface of revolution forming the wall of the cavity is assumed to have the form $r=f(x)$, where x is measured along the axis of the body and r perpendicular to it.

Assuming that a velocity potential satisfying the appropriate boundary conditions exists, the author derives from Green's theorem an integro-differential equation satisfied by $f(x)$. He then uses this to prove that, if $f(x)=x^k g(x)$ for some k such that $0<k<1$, where (1) $xg'(x)=o(g(x))$ $(x\to\infty)$, then k must be equal to $\frac{1}{2}$. Condition (1) is satisfied by algebraic combinations of $\log x$, $\log\log x$, etc., and implies that $g(x)$ lies between $x^{-\epsilon}$ and x^{ϵ} for large x for every $\epsilon>0$. Taking $k=\frac{1}{2}$, he shows that $\int^\infty x^{-1}g^2(x)dx$ cannot be finite and then that $(\log x)^{-\frac{1}{2}-\epsilon}<g(x)<(\log x)^{-\frac{1}{2}+\epsilon}$ for large x for every $\epsilon>0$. Under the further assumption that $g(x)=h(x)(\log x)^{-\frac{1}{2}}$, where (2) $(x\log x)h'(x)=O(h(x))$, the author shows finally that when x is large

$$f(x)=\frac{cx^{\frac{1}{2}}}{(\log x)^{\frac{1}{2}}}\left\{1-\frac{1}{8}\frac{\log\log x}{\log x}+O\left(\frac{1}{\log x}\right)\right\}.$$

Condition (2) is also satisfied by algebraic combinations of $\log x$, $\log\log x$, etc. The arguments used in the paper are essentially Tauberian in character. *F. Smithies.*

Ginsburg, I. P. On the theory of resistance of waves. Leningrad State Univ. Annals [Uchenye Zapiski] **87** [Math. Ser. **13**. Mechanics], 135–144 (1944). (Russian) [MF 16482]

In the present paper the author, using the method of the Fourier integral, gives a new derivation of a formula of Kochin [Transactions of the Conference on the Theory of Wave Resistance, Moscow, 1937]. The potential φ is a function which satisfies the equation $\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=0$ and the boundary conditions $\varphi_{zz}+k_0\varphi_z=0$ at $z=0$ and $\varphi_n=v_0\cos(n,x)$ along the boundary, where k_0 and v_0 are given constants and n the interior normal to the surface of the body. The author assumes the solution for φ in the form

$$\Re\left\{\frac{1}{2\pi}\int_B\int\sigma(x_1,y_1,z_1)ds\int_{-\pi}^{\pi}d\theta\int_0^\infty\{\exp[-k(z+z_1)+ik\omega]\right.$$
$$\left.+F(k,\theta)\exp[kz+ik\omega]\}dk\right\},$$

where $F(k,\theta)$ is an explicitly given function, B is the boundary of the body, ds is a surface element of B, $\omega=(x-x_1)\cos\theta+(y-y_1)\sin\theta$ and $\sigma(x_1,y_1,z_1)$ is a function which can be determined by solving a linear integral equation. Using certain transformations the author obtains the formula of Kochin. The resistance force is determined. The author shows that in the case when the immersed body is a sphere Kochin's formulae agree with those of Michell. *S. Bergman* (Cambridge, Mass.).

Kuo, Yung-Huai. The propagation of a spherical or a cylindrical wave of finite amplitude and the production of shock waves. Quart. Appl. Math. **4**, 349–360 (1947).

The notion of "limiting line" in the case of one-dimensional flows is well-known. This is a curve in the (x,t) plane (envelope of characteristics), across which an isentropic solution of the flow equations cannot be continuously extended. It is thus closely associated with the formation of a compressive shock. It had been surmised that in the case of expanding cylindrical or spherical waves the attenuation could be so great that shock formation might fail to take place. In this paper the author sets up the equations for the isentropic flows corresponding to these two geometries, with x, the radial coordinate, and t, the time, as independent variables; u, the radial velocity, and v, the enthalpy, as dependent variables. A solution (u,v) sets up a correspondence between the (x,t) and (u,v) planes which takes characteristics into characteristics, except at points where the Jacobian $J(u,v)$ of the transformation vanishes. The curve $J(u,v)=0$ is shown to correspond to the envelope l of one set of characteristics in the (x,t) plane, and l is called the "limiting line." When a path in the (x,t) plane nears the limiting line, acceleration and pressure gradient become infinite. The path has a cusp at l and turns back into the original region in a second branch. Thus the region is doubly covered and the solution is multiple-valued in the neighborhood of l. This situation is meaningless physically; the continuous solution must break down before l is reached.

The author then examines the so-called "lost solution" $J\equiv0$ and shows that there is a limiting line in this case too. In the last section an example is given in which it proves impossible to continue an isentropic solution across l either continuously or discontinuously (i.e., using l as a shock wave). To solve such a problem the condition of isentropy would have to be abandoned. *D. P. Ling.*

Kreyn, S. G. The behaviour of gasodynamic factors near the front of a striking wave. Rep. [Dopovidi] Acad. Sci. Ukrainian SSR no. 3–4, 11–16 (1946). (Ukrainian and English)

The author writes down the one-dimensional equations governing the transition of pressure, density and velocity across a normal shock wave ["striking wave" as mistranslated here] propagated into undisturbed gas. These quantities, taken just behind the shock, increase if the velocity D of propagation of the wave increases, as indeed is evident from inspection. The author then goes on to examine the signs of the gradients of these quantities just behind the shock. Owing to some oversight the signs are actually the reverse of those given. *D. P. Ling* (Murray Hill, N. J.).

***Ramponi, F. Nota sulla propagazione delle perturbazioni di regime nei canali aperti.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 493–502. Edizioni Cremonense, Rome, 1942.

Blinova, E. N. On the determination of the velocity of troughs employing the non-linear vorticity equation. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 669–670 (1946). (Russian. English summary)

Benel, Hilmi. Fréquences propres (partiels) d'un tuyau cylindrique fermé aux deux bouts, percé de trous latéraux égaux et équidistants. Rev. Fac. Sci. Univ. Istanbul (A) **11**, 47–78 (1946). (French. Turkish summary)

Elasticity, Plasticity

Prager, W. An introduction to the mathematical theory of plasticity. J. Appl. Phys. **18**, 375–383 (1947).

The paper aims at introducing the reader to the principal theories of plasticity. Since a presentation of the general stress-strain relations used in these theories would require too much space, the discussion is restricted to the mechanical behavior of plastic materials under shear. Theories of plastic deformation (Hencky, Nadai) and theories of plastic flow (Saint Venant-Lévy-Mises, Prandtl-Reuss, Prager) are illustrated by the example of a prismatic bar under torsion. *Author's summary.*

Il'yusin, A. A. Deformation of a visco-plastic solid. Učenye Zapiski Moskov. Gos. Univ. Matematika **39**, 1–81 (1940). (Russian) [MF 15365]

The differential equations for the plane flow of an incompressible isotropic visco-plastic material are derived under the assumption that flow occurs only when the maximum shearing stress exceeds a certain critical value, the maximum rate of shear being proportional to this excess. A variational principle is established which is equivalent to this system of differential equations and boundary conditions. The major part of the paper concerns the stability of visco-plastic flows with respect to small disturbances in the shape of the surface of the bodies under consideration. In particular, the uniform extension of a visco-plastic strip is discussed and its stability with respect to local contraction ("necking") is studied in detail. This investigation is extended to the expansion of cylindrical tubes by internal pressure. Finally, the compression of cylindrical samples by longitudinal blows is studied and theoretical and experimental results are compared with each other. The paper concludes with a brief discussion of the laws of mechanical similarity for visco-plastic materials. *W. Prager.*

Charrueau, André. Remarques sur l'équilibre d'un solide élastique, homogène et isotrope. Ann. Ponts Chaussées **1939 II** (109ᵉ année), 169–194 (1939).

Let N and T be the normal and shearing stresses corresponding to a direction d at a point P. As d is varied, the point Q with rectangular Cartesian coordinates (N, T) moves in a closed region bounded by three circles which are tangent in pairs and have collinear centers. These circles are called the Mohr circles. Their centers and radii depend only on the principal stresses N_1, N_2 and N_3 at the point P. According to Mohr's theory, plastic deformation occurs at those points in a stressed body for which the largest Mohr circle is tangent to a certain curve in the (N, T)-plane, called the intrinsic curve. Corresponding to a given intrinsic curve there are then relations between N_1, N_2 and N_3, which are satisfied wherever there is plastic deformation. If now N_1, N_2 and N_3 are treated as rectangular Cartesian coordinates, these relations define surfaces which enclose a region V. The present paper is concerned with the determination of the region V corresponding to a given intrinsic curve. In particular, the cases when the intrinsic curve is a straight line or a semi-cubical parabola are treated in some detail. Finally, the author considers the problem of finding the surface of separation of the elastic and plastic regions.
G. E. Hay (Ann Arbor, Mich.).

Charrueau, André. Sur une transformation géométrique utilisée dans l'étude de l'équilibre d'un solide élastique. Ann. Ponts Chaussées **1940 I** (110ᵉ année), 127–146 (1940). [MF 12879]

This is a continuation of the paper reviewed above. The case is considered when N_1 and N_2 are, respectively, the largest and smallest of the principal stresses. In this case the region V is a cylinder with generators parallel to the N_3-axis. The paper deals primarily with properties of the transformation which maps a given intrinsic curve onto the (N_1, N_2)-trace of this cylinder. For example, it is shown that this transformation is a contact transformation which maps straight lines onto straight lines and conics onto conics.
G. E. Hay (Ann Arbor, Mich.).

Charrueau, André. Remarques sur la résistance élastique des corps isotropes. Ann. Ponts Chaussées **1941 I** (111ᵉ année), 215–221 (1941). [MF 12882]

According to the Mohr theory, the condition for plastic deformation involves only the largest of the Mohr circles, which depends only on the largest and the smallest of the principal stresses. In the present paper the author discusses the more general case when the condition for plastic flow depends on all three of the principal stresses.
G. E. Hay (Ann Arbor, Mich.).

Charrueau, André. Équilibres limites de certains milieux indéfinis dans le cas d'une courbe intrinsèque quelconque. Ann. Ponts Chaussées **1940 II** (110ᵉ année), 203–222 (1940). [MF 12880]

The author considers an infinite wedge. The load per unit area on one face is a known constant and on the other face is an unknown constant. It is assumed that the equilibrium is plastic everywhere and that the stress components are independent of the distance from the edge of the wedge. The stress components are determined in the case of a

general intrinsic curve. It is found that there are two types of solution. As an example, the case is considered when the angle between the faces of the wedge is π and the equation of the intrinsic curve is $y^n = ax+b$, $n \geqq 1$, $a>0$. One type of solution applies in regions near the external boundaries and the other type applies elsewhere, the boundaries between these regions being planes through the edge of the wedge.
G. E. Hay (Ann Arbor, Mich.).

Charrueau, André. Sur les équilibres limites plans des milieux continus isotropes. Ann. Ponts Chaussées **1944** (114e année), 315–324 (1944). [MF 13964]

The author first presents some additional comments on the contact transformation mentioned in the second paper of the sequence reviewed above. The rest of the paper is devoted to a study of plane plastic deformation. The stress components are denoted by N_x, N_y, T_{xy}. The Mohr yield condition is expressed in the form $f(N_x, T_{xy}, N_y)=0$. The stress components are regarded as rectangular Cartesian coordinates, whence this equation defines a surface Σ. Some geometrical properties of Σ are discussed. The problem in the paper reviewed just above is then considered again and a solution based on geometrical construction is presented. The plane plastic deformation of a body which is isotropic but is heterogeneous is also considered. The slip lines are determined as the characteristic curves of two partial differential equations. *G. E. Hay* (Ann Arbor, Mich.).

***Cisotti, Umberto. Formule integrali relative alla meccanica dei sistemi continui.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 404–410. Edizioni Cremonense, Rome, 1942.

Tonolo, A. Contributo alla teoria dell'elasticità dei corpi solidi. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **100**, 383–395 (1941).

Somigliana a donné les équations de l'équilibre d'un corps élastique, dans l'hypothèse que sur chaque élément matériel agit une force de masse et un couple. Ces équations étant exprimées en coordonnées cartésiennes, l'auteur développe la transformation en coordonnées curvilignes en se servant du calcul différentiel absolu. *B. Levi* (Rosario).

***Signorini, Antonio. Deformazioni elastiche finite: elasticità di 2° grado.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 56–71. Edizioni Cremonense, Rome, 1942.

Expository lecture.

***Gran Olsson, R. The principle of virtual displacements applied in approximate solutions of eigenvalue problems.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 255–258. Jul. Gjellerups Forlag, Copenhagen, 1947.

Lazard, A. Flambement en milieu élastique discontinu. Efficacité d'un dispositif de contre-flambage. Ann. Ponts Chaussées **1946** (116e année), 289–329 (1946).

A uniform column with elastic lateral supports is considered. The critical loads for various end conditions and support distributions are obtained. The optimum support stiffnesses are discussed. *G. F. Carrier*.

Stiles, W. B. Bending of clamped plates. J. Appl. Mech. **14**, A-55–A-62 (1947).

The Weinstein technique, where one replaces a given boundary value problem by a sequence of simpler boundary value problems, is used to determine the deflection of a rectangular elastic plate under various loadings and conditions of support. The results compare reasonably well with experiment but it is not obvious whether this method is less laborious than the more classical procedures.
G. F. Carrier (Providence, R. I.).

***Zanaboni, Osvaldo. Soluzione caratteristica della lastra rettangolare a due lati appoggiati, sotto l'azione di forze e coppie concentrate.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 745–752. Edizioni Cremonense, Rome, 1942.

***Zanaboni, Osvaldo. Soluzione della lastra rettangolare sotto carichi comunque distribuiti lungo linee e superficie.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 753–762. Edizioni Cremonense, Rome, 1942.

***Faedo, Sandro. Deformazione di una piastra a spessore variabile soggetta a pressione.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 205–209. Edizioni Cremonense, Rome, 1942.

***Agostinelli, Cataldo. Vibrazioni e pressioni critiche in una piastra circolare sollecitata al contorno da una pressione radiale.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 360–370. Edizioni Cremonense, Rome, 1942.

Seth, B. R. Transverse vibrations of rectilinear plates. Proc. Indian Acad. Sci., Sect. A. **25**, 25–29 (1947).

Several similar problems in the transverse vibrations of plates with rectilinear edges are investigated. Results are given for various triangular plates which are simply supported. Erroneous results are given for the vibrations of a square plate with clamped edges. *G. F. Carrier*.

Bellin, A. I. Determination of the natural frequencies of the bending vibrations of beams. J. Appl. Mech. **14**, A-1–A-6 (1947).

This paper presents a method for determining the natural frequencies of lateral vibrations for elastic beams of variable cross section and any number of spans.
From the author's summary.

Shapiro, G. S. Longitudinal oscillations of bars. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 597–616 (1946). (Russian. English summary)

The paper contains a careful analysis of the mathematical problems arising in the study of longitudinal oscillations of elastic-plastic bars. Numerous examples are treated by the method of characteristics. *W. Prager*.

Krall, G. Vibrazioni di uno scafo elastico galleggiante su un fluido sede di propagazioni ondose. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 937–948, 1272–1280 (1946).

Cattaneo, C. Su un teorema fondamentale nella teoria delle onde di discontinuità. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 66–72 (1946).

Dans cette première note l'auteur, après avoir rappelé une objection de Duhem à la démonstration d'un théorème d'Hadamard, selon lequel les conditions de stabilité de l'équilibre interne d'un corps élastique assurent que la quadrique de polarisation est un ellipsoide, expose de nouveau le problème dans l'hypothèse d'appliquer une récente généralisation des équations de l'élasticité due à Signorini. *B. Levi* (Rosario).

Cattaneo, C. Su un teorema fondamentale nella teoria delle onde di discontinuità. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 728–734 (1946).

[Voir l'analyse ci-dessus.] Cette note fait suite à la première dont elle conserve les notations et la numération des formules. On y considère en détail la démonstration d'Hadamard et on propose une réalisation concrète des déplacements dans une petite région du corps, qui assure la validité du raisonnement de cet auteur. *B. Levi* (Rosario).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Courvoisier, L. Über die theoretische Ungleichheit von Reflexionswinkel und Einfallswinkel am bewegten Spiegel. Verh. Naturforsch. Ges. Basel 57, 25–29 (1946).

The author derives an approximate relation between the angle of incidence and the angle of reflection from a moving mirror. This approximate relation is shown to agree with the relations derived by A. Harnack [Ann. Physik (4) **39**, 1053–1058 (1912)] up to third order terms in $\beta=v'/c$, where v' is the component of the velocity of the mirror in the direction of its normal. The author then discusses an application of these formulas. *A. H. Taub* (Princeton, N. J.).

Lansraux, Guy. Calcul des figures de diffraction des pupilles de révolution. Rev. Optique 26, 24–45 (1947).

The paper discusses the effects on the axial image formed by a system of revolution-symmetry which result when radially symmetrical changes are made in the amplitude and phase of the waves leaving the exit-pupil. Defocusing and primary spherical aberration at various focal adjustments are covered as special cases. The method adopted is to expand the wave-function $F(\rho)$ which specifies the complex displacement in the (spherical) reference-surface filling the exit-pupil $0\leqq\rho\leqq R$ as a Taylor series in powers of ρ^2/R^2-1 and to show that, if $F(\rho)=\sum_{p=1}^{\infty}a_p(1-\rho^2/R^2)^{p-1}$, then the complex displacement $G(r)$ at a point P near the axis in the image-plane is given by the equation $G(r)/(\pi R^2)=\Gamma(W)=\sum_{p=1}^{\infty}p^{-1}a_pL_p(W)$, where

$$L_p(W)=2^p p!J_p(W)/W^p,$$

J_p is the Bessel function of order p, $W=2\pi Rr$, R is the radius of the exit-pupil in wave-lengths, r the sine of the off-axis angle of P subtended at the centre of the exit-pupil.

Computational results are obtained for the intensities in the image-plane (1) of an error-free extrafocal image, (2) of the image by a system of variable transmission $e^{-k\rho^2/R^2}$, (3) of a system suffering from combined defocusing and spherical aberration. In case (2) it is shown that the Airy rings disappear for all practical purposes when $k=4$, leaving only a central peak which is not much broader than that of the Airy disc. [There is an inconsistency in notation in this section of the paper.] A valuable feature is the estimation of the error of approximation to $\Gamma(W)$ which may result from a given error of approximation to $F(\rho)$. [The author seems to be unaware of previous work on the subject, in particular, of an investigation by A. E. Conrady, Monthly Not. Roy. Astr. Soc. **79**, 575–593 (1919).] *E. H. Linfoot* (Bristol).

Ordered

***Brillouin, Léon. Wave Propagation in Periodic Structures. Electric Filters and Crystal Lattices.** McGraw-Hill Book Company, Inc., New York, 1946. xii+247 pp. $4.00.

The first four chapters [70 pp., approximately] deal with wave propagation along infinite one-dimensional lattices. Boundary conditions arising in finite lattices, or in the junction of different lattices, are discussed in chapter V, in terms of energy velocity, energy flow, characteristic impedance and the like [25 pp.]. Chapters VI and VII are devoted to wave propagation along lattices in two and three dimensions [80 pp.]. Chapter VIII gives a survey of Mathieu's equation and related problems [20 pp.]. The last two chapters discuss the propagation of waves along electric lines [50 pp.].

Chapter I refers to early work of Newton, Bernoulli, Euler, Lagrange and others, much of which cleared the way for numerous problems in modern physics and mathematics, Kelvin's theory of dispersion (rediscovered by Born in 1912), as perfection and generalization of Cauchy's and Baden-Powell's theories (cutoff frequency, low-pass, high-pass and band-pass filters, acoustical and optical branches of the dispersion curve). Chapter II contains some general results and a qualitative discussion for the case of vibrations in rows of monatomic and diatomic molecules, the mathematical theory of which is given in chapters III (identical particles, interaction between nearest neighbors only, discussion of the distance of interaction, analogies between electrical and mechanical systems) and IV (more complicated one-dimensional lattices, thorough discussion of the linear NaCl lattice, transition from diatomic to monatomic lattices, polyatomic molecules). [It is somewhat confusing that the signs of wavenumber (a) and wavelength (λ) are not always clear; sometimes $a=1/\lambda$, again $|a|=1/\lambda$; see equations (4.2) and (4.7). In the reviewer's opinion the total potential energy discussed in § 8 is not finite; (8.6b) may be better derived from the energy of the pth particle. In equation (10.1) one should take $0<m\leqq L$ to make the text on p. 34 consistent; ν^2, rather than ν, is a polynomial of degree L, and there are L solutions in the interval 0 to $1/2d$.]

Chapters VI and VII, not as elementary as the preceding ones, contain much that was first developed by the author in his original treatises (theory of zones, connection between stopping bands and selective reflection); cf. L. Brillouin, Quantenstatistik, Springer, Berlin, 1931. The general discussion of the zone theory given here will serve [as suggested by the author] as an introduction to other textbooks relating to the theory of solids [N. F. Mott and H. Jones, The Theory of the Properties of Metals and Alloys, Oxford, 1936; F. Seitz, The Modern Theory of Solids, McGraw-Hill, New York, 1940]. Chapters VIII and IX treat the usual properties of four-terminal networks; cascade circuits in connection with the group C of two-rowed complex matrices of unit determinant, which is an integral part of the theory of the electron spin; Pauli matrices in relation to continuous electrical lines. *C. J. Bouwkamp* (Eindhoven).

Carlson, J. F., and Heins, A. E. The reflection of an electromagnetic plane wave by an infinite set of plates. I. Quart. Appl. Math. 4, 313–329 (1947).

In some unpublished work on the theory of guided waves, J. S. Schwinger has shown that a special class of boundary value problems in electrodynamics can be formulated mathematically as Wiener-Hopf integral equations. In this class

of problem, a plane wave is incident upon a number of perfectly conducting parallel half-planes with parallel edges, the wave being so polarised that the problem is two-dimensional. It is then possible to express the electric and magnetic vectors in terms of the surface current density on the plates and the vanishing of the tangential components of the electric vector leads to a system of integral equations for the various surface current densities. The present paper deals with the case when the reflecting structure has a certain periodicity, which reduces the system of integral equations to a single equation. The half-planes are equally spaced and equally staggered, so that, on them, $x=na$, $z\geqq nb$ ($n=0, \pm 1, \pm 2, \cdots$). The electric force $\varphi_0(x, z)$ in the incident field is $e^{ik(x\cos\theta+z\sin\theta)}$ (a time factor e^{-ikct} being understood), in the direction of Oy.

It is readily seen that the total electric force is also parallel to Oy and is given by an equation

$$\varphi(x, z)=\varphi_0(x, z)$$
$$+\tfrac{1}{4}i\sum_{-\infty}^{\infty}\int_{mb}^{\infty} I_m(z')H_0^{(1)}(k\{(z-z')^2+(x-ma)^2\}^{\frac{1}{2}})dz',$$

where I_m denotes the current density in the mth plate and $H_0^{(1)}$ denotes the Bessel function of the third kind; the system of integral equations is obtained by making φ vanish on each plate. The symmetry of the structure is readily shown to reduce this rather complicated system to the single equation

$$e^{ikz\cos\theta}=-\tfrac{1}{4}i\sum_{-\infty}^{\infty}\int_{0}^{\infty} I_0(z')e^{ikq(a\sin\theta+b\cos\theta)}$$
$$\times H_0^{(1)}(k\{q^2a^2+(gb+z-z')^2\}^{\frac{1}{2}})dz'$$

when $z>0$. The authors solve this equation by the Wiener-Hopf method, the result being rather too complicated to quote here. The method gives in fact, not $I_0(z)$, but its complex Fourier transform. This is no disadvantage, since the formula for $\varphi(x, z)$ can also be written in a form involving the Fourier transform of $I_0(z)$; the authors determine the form of $\varphi(x, z)$ for $|z|$ large, and hence the physically important reflection and transmission coefficients. The periodic structure has some properties which are analogous to those of metal mirrors and gratings; thus when it is excited by a plane wave, there are reflected plane waves in certain directions depending on the relative dimensions, the wave-length and the direction of incidence.

E. T. Copson (Dundee).

Feld, J. N. Radiating slit systems. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 615–618 (1946).

In an earlier paper [same C. R. (N.S.) **51**, 115–118 (1946); these Rev. **7**, 534] the author discussed radiation through narrow slits in systems with axial symmetry. In the present paper he considers a single asymmetrical slit in a resonator which is a perfect conductor in the form of a surface of revolution. Thus it is no longer assumed that the potential difference between the edges of the slit is constant along the slit. The field inside the resonator is excited by a linear conductor so adjusted that the induced current on the uncut resonator over the slit region is approximately perpendicular to the edges of the slit. The author obtains an integral equation for the potential difference across the slit; this is solved by a process of successive approximations closely related to that used in recent work on antenna theory. He concludes that the radiation from the slit is very similar to that from a suitably excited metallic antenna of the same dimensions.

M. C. Gray (New York, N. Y.).

Kisunko, G. V. On the theory of the excitation of radio-wave guides. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 199–202 (1946).

A formal complete solution of the straight cylindrical wave guide problem is given. It is obtained in terms of the eigenfunctions of the two-dimensional Dirichlet and Neumann interior problems for the simply connected guide contour S, i.e., of the characteristic functions Φ_n or Φ_m of a flat membrane with circumference S and boundary conditions $\Phi_n=0$ or $\partial\Phi_m/\partial\mathbf{n}=0$, respectively. The field vectors $\mathbf{E}$ and $\mathbf{H}$ of the guide extending in the z-direction are obtained in the following form, representing superpositions of the well-known E(TM) and M(TE) wave types:

$$\mathbf{E}=\sum_{(n)}(p_{ln}\mathbf{L}_n+p_{gn}\mathbf{G}_n)+\sum_{(m)}p_{cm}\mathbf{C}_m,$$
$$\mathbf{H}=\sum_{(n)}q_{cn}\mathbf{C}_n+\sum_{(m)}(q_{lm}\mathbf{L}_m+q_{gm}\mathbf{G}_m),$$

with $\mathbf{L}_m\equiv\Phi_m\cdot\zeta$ (ζ, unit z-vector),

$$\mathbf{G}_m\equiv\kappa_m^{-1}\nabla\cdot\Phi_m,\quad \mathbf{C}_n=\kappa_n^{-1}\nabla\times(\Phi_n\zeta),\quad (\nabla_2^2+\kappa_{n;m}^2)\phi_{n;m}=0$$

and

$$p_{ln}=-c^{-1}\dot{a}_{ln}-\partial\varphi_n/\partial z,\quad p_{gn}=-c^{-1}\dot{a}_{gn}-\kappa_n\phi_n,$$
$$q_{cn}=\kappa_n a_{ln}-\partial a_{gn}/\partial z,\quad q_{lm}=a_{cm}\kappa_m,$$
$$q_{gm}=\partial a_{cm}/\partial z,\quad p_{cm}=-c^{-1}\dot{a}_{cm},$$

where the a_r and φ_n are the expansion coefficients of the vector and scalar potentials

$$\mathbf{A}=\sum_{(r)}a_r(z, t)\mathbf{A}_r(\xi, \eta),\quad \Phi=\sum_{(n)}\varphi_n(z, t)\Phi_n(\xi, \eta)$$

(with the orthogonal coordinates (ξ, η) obtained from conformal mapping of S) and fulfill the continuity equation $\partial a_{ln}/\partial z-\kappa_n a_{gn}+c^{-1}\dot{\varphi}_n=0$ and the scalar wave equations

$$\left(\frac{\partial^2}{\partial z^2}-c^{-2}\frac{\partial^2}{\partial t^2}-\kappa_r^2\right)a_r=-4\pi c^{-1}j_r,\quad j_r(z, t)=\int_{(S)}\mathbf{j}\mathbf{A}_r^*d\sigma,$$
$$\left(\frac{\partial^2}{\partial z^2}-c^{-2}\frac{\partial^2}{\partial t^2}-\kappa_n^2\right)\varphi_n=-4\pi c^{-1}\rho_n,\quad \rho_n(z, t)=\int_{(S)}\rho\Phi_n^*d\sigma.$$

For harmonic time dependence, only the inhomogeneous solutions of these equations apply in the case of an infinitely long guide in the absence of passive conductors, by virtue of the radiation principle. If passive conductors are present, which may be the case in a certain finite segment, these mode amplitudes are written as Fourier integrals over the current and charge distribution along these segments.

H. G. Baerwald (Cleveland, Ohio).

Krasnooshkin, P. Acoustic and electromagnetic wave guides of complicated shape. Acad. Sci. USSR. J. Phys. **10**, 434–445 (1 plate) (1946).

The author discusses the propagation of electromagnetic and acoustical waves in hollow curved pipes of variable rectangular cross-section. These pipes are formed by two parallel planes and two nonintersecting cylindrical surfaces normal to these planes. The problem can therefore be reduced to the consideration of the two-dimensional wave equation. The variations in shape and cross-section are further limited to those plane figures which can be transformed by an analytic function into the longitudinal section of a uniform straight pipe. Finally it is assumed that the wave equation is separable in these transformed variables.

The boundary conditions at the edges of the pipe together with one of the resulting ordinary differential equations form a Sturm-Liouville eigenvalue problem. Use is made of this fact in applying the Courant minimax principle to obtain estimates of the eigenvalues. The parabolic, elliptic and toroidal pipes are treated in detail and are, respectively, shown to illustrate tunneling, band-pass properties and clinging phenomena. *R. S. Phillips* (New York, N. Y.).

Basile, Stefano. **La propagazione delle onde elettromagnetiche lungo tubi.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 803–813. Edizioni Cremonense, Rome, 1942.

van der Pol, Balth. The fundamental principles of frequency modulation. J. Inst. Elec. Engrs. Part III. **93**, 153–158 (1946).

Lecture to the Radio Section of the Institution.

Agostinelli, Cataldo. **Nuovo procedimento di integrazione per serie delle equazioni del moto di un corpuscolo elettrizzato in presenza di un dipolo magnetico.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 371–381. Edizioni Cremonense, Rome, 1942.

Takahasi, Hidetosi. Über die Beziehungen zwischen dem reellen und dem imaginären Teil einer frequenzabhängigen komplexen Grösse. Proc. Phys.-Math. Soc. Japan (3) **24**, 63–75 (1942). [MF 15020]

Takahasi, Hidetosi. Das Problem der Stabilisierung. Proc. Phys.-Math. Soc. Japan (3) **24**, 412–433 (1942). [MF 15030]

H. A. Kramers in 1925 discovered a relation between absorption and dispersion of X-rays by atoms, in the form of an integral transformation

$$\xi(\omega)=(2/\pi)\int_0^\infty(\omega^2-\lambda^2)^{-1}\lambda\eta(\lambda)d\lambda,$$

or its inverse, $\xi+i\eta$ being the complex dielectric constant, ω the frequency [Atti Congresso Internaz. Fisici, Como, 1927, vol. 2, pp. 545–557]. He also pointed out that this formula is related to well-known results in the theory of functions expressing a regular function in terms of its boundary values (Dirichlet problem for the half-plane). The same relationship was also found, though in less general terms, by R. de L. Kronig [J. Opt. Soc. Amer. **12**, 547–557 (1926)] and Kallmann and Mark [Ann. Physik (4) **82**, 585–604 (1927)]. In the meantime, in the field of electrical engineering J. R. Carson had recognized in principle [Electric Circuit Theory, McGraw-Hill, New York, 1926] the existence of such a relation between the real and imaginary parts of the impedance of a network, but it was not until 1935 that the formula of Kramers was explicitly found by Bayard [Rev. Gén. Électricité **37**, 659–664 (1935)] for a Kirchhoff two-pole. Exactly corresponding relations are also found in acoustics.

Then B. Gross [Physical Rev. (2) **59**, 748–750 (1941)] and H. Takahasi [the first paper under review] independently pointed out the generality of the relation under discussion, Gross giving a common derivation for anomalous dielectrics and electric networks, and Takahasi for very general linear dynamical systems. [See also B. Gross, Amer. Math. Monthly **50**, 90–93 (1943); these Rev. **4**, 98; H. W. Bode, Network Analysis and Feedback Amplifier Design, Van Nostrand, New York, 1945, chaps. 14, 15.]

The formula of Kramers is well known to physicists, but Takahasi states that he did not have access to the original paper, and it is apparent also that numerous other authors in this field did not appreciate fully the implications of Kramers' first and fundamental paper. In the second paper under review, Takahasi considers the problem of making a given linear labile system stable by coupling to it a second system. He uses as one of his tools the results of the first paper, restated in the form of necessary and sufficient conditions on the impedance functions of the two systems. Methods of solution by means of Lagrangian interpolation formulas are discussed, and applications are made to arc lights (negative resistance), to amplifiers and to a particular electromechanical system. *L. C. Hutchinson.*

Cheng, Tseng-Tung. The limit of certain matrices. Coll. Papers Sci. Engin. Nat. Univ. Amoy **1**, 55–64 (1943).

A uniform continuous transmission line may be thought of as the limit of nT or π sections as n increases indefinitely and the series impedance and shunt admittance go to zero as $1/n$. This paper verifies that the direct multiplication of the corresponding matrices yields in the limit the same result as the usual simpler derivation. *L. C. Hutchinson.*

van Slooten, Jacob. **Meetkundige Beschouwingen in Verband met de Theorie der Electrische Vierpolen. [Geometrical Considerations in Connection with the Theory of Four-Terminal Networks].** Thesis, Technische Hoogeschool te Delft, 1946. 87 pp. (Dutch. English summary)

A four-terminal network can be interpreted as an impedance transformer by considering the impedance Z_1 appearing between the input terminals as a function (transformation) of the impedance Z_2 connected to the output terminals. This transformation corresponds to a general linear transformation of the complex Z-plane. The so-called iterative impedances of the network are mapped on the invariant points of the linear transformation. In the first two chapters the geometrical properties of the transformation are treated and some applications to the theory of four-terminal networks are discussed. The major problem of the remaining three chapters is the following. When two networks of known transformation properties are connected in cascade (in parallel), what are the transformation properties of the resulting network? Algebraically this reduces to properties of matrix multiplication (addition) but the algebraic method does not lead to simple results. A geometric study of the problem has been made by A. Weissfloch [Hochfrequenztech. Elektroak. **61**, 19–28, 100–123 (1943); these Rev. **5**, 164] but, in the author's opinion, the constructions involved are quite complicated. It is furthermore pointed out that a motion of three-dimensional non-Euclidean space corresponds to each transformation by a four-terminal network. For resistanceless networks this reduces to a motion of the non-Euclidean plane. Accordingly, a connection in cascade is equivalent to a superposition of non-Euclidean motions. In the fourth chapter a simple geometric construction for this superposition is worked out in case of plane motions, not with reference to Poincaré's model of non-Euclidean geometry, but rather to that of Cayley. This construction, which is applied to transmission lines in the last chapter, can be interpreted as an addition of sliding vectors. There are an extensive seven-page English summary and a bibliography of 31 items. *C. J. Bouwkamp* (Eindhoven).

Tellegen, B. D. H. Network synthesis, especially the synthesis of resistanceless four-terminal networks. Philips Research Rep. 1, 169–184 (1946).

The following is an extract from the author's summary. In considering two-terminal networks the concept of order is introduced and defined as the order of the differential equation of the free vibrations of the system which is formed by connecting the terminals over a resistance. By extending the concept of order to four-terminal networks it is possible to indicate the resistanceless four-terminal networks of a given order. It is found that four different four-terminal networks of odd order are possible and five of even order.
A. L. Foster (Berkeley, Calif.).

Quade, W. Matrizenrechnung und elektrische Netze. Arch. Elektrotechnik 34, 545–567 (1940).

This is an expository paper on the application of matrices and linear graphs to electrical network theory. The first half is a self-contained introduction to the matrix calculus. The remainder of the paper develops the theory of networks, summarizing the author's own work on invariant properties [Klassifikation der Schwingungsvorgänge in gekoppelten Stromkreisen, Leipzig, 1933] as well as the familiar work of Cauer, Feldtkeller, Ingram and others in this field. *L. C. Hutchinson* (Brooklyn, N. Y.).

Quantum Mechanics

Bodiou, Georges. Démonstration du principe d'ondulisation de la mécanique quantique à partir d'un postulat statistique. C. R. Acad. Sci. Paris 223, 848–850 (1946).

If a quantum mechanical state is represented in configuration space by the function $(q|)$ and in momentum space by the function $(p|)$, then

$$\text{(A)}\qquad (p|)=\int_{-\infty}^{\infty} h^{-\frac{1}{2}} \exp(-2\pi i h^{-1}pq)(q|)dq.$$

By (A) a Gaussian distribution $|(q|)|^2$ is transformed into a Gaussian distribution $|(p|)|^2$. The author proves that this property is characteristic of the transformation (A). Let

$$\text{(B)}\qquad (p|)=\int f(p,q)(q|)dq.$$

If (B) transforms every Gaussian distribution $|(q|)|^2$ (possibly after $(q|)$ has been multiplied by an appropriate phase factor $\exp iF(q)$) into a Gaussian distribution $|(p|)|^2$ whose mean value and dispersion are independent of the mean value $\bar{q}$, then $f(p,q)$ is necessarily of the form $h^{-\frac{1}{2}}\exp(-2\pi i h^{-1}pq)$ with some constant h. The author proposes to base quantum mechanics on this theorem.
V. Bargmann (Princeton, N. J.).

Corson, E. M. The invariant form of quantum equations and the Schroedinger-Heisenberg parallelism. Physical Rev. (2) 71, 200–208 (1947).

Familiar ideas in the transformation theory of quantum mechanics and in the equations of motion are discussed with particular reference to the relationship between the Schrödinger and Heisenberg representations, which is illustrated by diagrams. A form of the equations of motion is given valid for both representations and also in a form involving the density operator. *C. Strachan* (Aberdeen).

Ma, S. T. On a general condition of Heisenberg for the S matrix. Physical Rev. (2) 71, 195–200 (1947).

Heisenberg has shown [Z. Physik 120, 513–538, 673–702 (1943); these Rev. 4, 292] that the S matrix which plays a fundamental role in his new formulation of quantum theory is related to the asymptotic behavior of the wave functions by the following relation for large values of r:

$$(*)\qquad \int_{-\infty}^{\infty} S(k)\exp(ikr)dk=\sum_n |c_n|^2 \exp(-|k_n|r),$$

where the discrete wave functions have the asymptotic behavior $\mu_n(r)\sim c_n(2\pi)^{-\frac{1}{2}}\exp(-|k_n|r)$. The author shows that this equation is not valid for a system in which an exponential attractive potential exists. The validity of (*) is ensured if the supplementary condition that the interaction potential vanish at large distances is introduced. It is shown that such a cut-off at a distance R together with the use of the S matrix gives the correct eigenvalues of the energy and asymptotic behavior of the wave functions for S states in the limit $R\to\infty$ for two cases: the attractive exponential potential and the Coulomb potential.
A. H. Taub (Princeton, N. J.).

Dallaporta, N. Somma delle approssimazioni successive del metodo di Born nella teoria degli urti. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 102, 443–454 (1943).

Dallaporta, N. Somma delle approssimazioni successive del metodo di Born nella teoria degli urti. II. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 102, 519–530 (1943).

The author sets himself the task of completing the work of Møller [Z. Physik 66, 513–532 (1930)] and Distel [Z. Physik 74, 785–809 (1932)] on the higher Born approximations. The first paper deals with elastic collisions, the second with inelastic ones. The formula for the elastic scattering cross section derived by the author [equations (24), (24a), p. 453] is surprisingly simple but can be disproved as far as the second and higher approximations are concerned.
L. Hulthén (Lund).

★Hulthén, Lamek. The variational principle for continuous spectra. C. R. Dixième Congrès Math. Scandinaves 1946, pp. 201–206. Jul. Gjellerups Forlag, Copenhagen, 1947.

Consider the continuous spectrum of the Schroedinger equation $D(E,l)\psi(r)=0$ for the radial part $\psi(r)$ of the wave function of relative motion in a spherically symmetric two-body problem (E, energy; l, azimuthal quantum number). It is shown that $\int_0^\infty \psi(r)D(E,l)\psi(r)dr$ is stationary for a variation $\delta\eta(E,l)$ of the phase $\eta(E,l)$ of $\psi(r)$ at $r\to\infty$; E and l are kept constant in varying η. Using this stationarity the phase can be computed by variational methods. Numerical results are given for a relative potential of the type e^{-ar}/r.
A. Pais (Princeton, N. J.).

Drăganu, Mircea. Sur l'équation de Schrödinger en coordonées quelconques. Disquisit. Math. Phys. 5, 115–121 (1946).

Bopp, Fritz. Quantentheorie der Feldmechanik. Z. Naturforschung 1, 196–203 (1946).

The classical field equation previously derived by the author [Ann. Physik (5) 38, 345–384 (1940); 42, 573–608 (1942); these Rev. 2, 336; 8, 124] is considered in the

approximation where the radiation reaction is neglected. The Lagrangian function for a single particle with coordinates $x_\alpha(s)$, $u_\alpha=\dot{x}_\alpha(s)$, $\alpha=1, \cdots, 4$, in an external field described by the four-potential A_α has the form

$$\int L ds = e^2c^{-2}\int u_\alpha(s)u_\alpha(s')f(\sigma)dsds' - ec^{-1}\int u_\alpha A_\alpha ds,$$

where $\sigma=c^{-2}\{x_\alpha(s)-x_\alpha(s')\}^2$. Now s' is replaced by $s-\tau$ and the Lagrangian L is developed in powers of the retardation parameter τ. The first approximation gives the usual relativistic equation of motion for a charged particle and the next approximation gives a Lagrangian function which involves $\dot{u}_\alpha$. This is interpreted as giving rise to a new degree of freedom. Using the method of Whittaker, the author makes a Hamiltonian theory for this case and it is shown that the new degree of freedom gives rise to a spin-like motion of the particle. Translation to quantum theory is made by demanding canonical commutation relations for the conjugate variables; expressions for the wave equation, charge-current vector and energy-momentum tensor are obtained. The wave equation is studied for the simplest case of a particle with no linear momentum and no external force and it is shown that its eigenfunctions give rise to states with different masses and spin values. These states do not correspond to actual particles in nature quantitatively, but the author states that such deviations are to be expected from the approximations made in the theory.

S. Kusaka (Princeton, N. J.).

Bopp, Fritz. Der Energie-Impuls-Tensor in einer Fernwirkungsfeldtheorie. Z. Naturforschung 1, 237–242 (1946).

In a field theory in which the Lagrangian density is a function of two points, $L_0=\frac{1}{2}\varphi(x)\epsilon_0(x-x')\varphi(x')$, the usual method of deriving the energy-momentum tensor is not applicable. In this paper the author generalizes the usual procedure so that the tensor can be derived for this more general case. His method is to consider the field variables to be functions of both coordinates, $\varphi=\varphi(x, x')$, $\varphi'=\varphi(x', x)$, which, however, satisfy the supplementary conditions $\partial\varphi/\partial x_\alpha'=0$, $\partial\varphi'/\partial x_\alpha=0$. Then the Lagrange method of undetermined multipliers is used so that in the new Lagrangian, $L=\frac{1}{2}\varphi\epsilon_0\varphi'+\frac{1}{2}(\Lambda_\alpha\partial\varphi'/\partial x_\alpha+\Lambda_\alpha'\partial\varphi/\partial x_\alpha')$, the variational derivatives with respect to φ and Λ can be taken independently. In this form the usual method of deriving the energy-momentum tensor can be applied. As a simple example, the special case of $\epsilon_0=(\kappa^2-\Box)\delta(x-x')$ in which the wave field satisfies the ordinary Klein-Gordon equation $(\Box-\kappa^2)\varphi=0$ is considered and it is shown that the energy-momentum tensor derived in this way differs from the usual expression by a term which can be written as a divergence of a tensor of the third rank. It is then shown that, by using the function $\epsilon(x)$ which satisfies the equation $\Box\epsilon(x)=-\epsilon_0(x)$ and writing the Lagrangian in the form

$$L_0=\frac{1}{2}\frac{\partial\varphi(x)}{\partial x_\alpha}\epsilon(x-x')\frac{\partial\varphi(x')}{\partial x_\alpha'},$$

the same procedure leads to a tensor of the usual form.

S. Kusaka (Princeton, N. J.).

Chang, T. S. A note on the Hamiltonian theory of quantization. II. Proc. Cambridge Philos. Soc. 43, 196–204 (1947).

This paper is a continuation of the author's previous work [Proc. Roy. Soc. London. Ser. A. 183, 316–328 (1945); these Rev. 6, 224] and extends some of the results obtained there. In the first part of the paper it is pointed out that the equations of motion for any field obtained by varying a Lagrangian subject to auxiliary conditions are exactly equivalent to a certain set of canonical equations and that on passing to quantum theory the commutation relations between the dynamical variables are Lorentz invariant. It is next shown that a similar procedure can be applied to the case of missing momenta and that the equations can be put in canonical form if a certain condition is satisfied, but that the commutation relations are in general not Lorentz invariant. However, in cases where the Lagrangian satisfies the condition of being "gauge invariant," then some of the momenta must be missing and the corresponding Eulerian equations can be replaced by equations containing only the momenta and the commutation relations between the variables are Lorentz invariant.

S. Kusaka.

Markov, M. A. On the criterion of relativistic invariance. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 16, 790–799 (1946). (Russian. English summary)

[Reviewer's note: the Russian title should be translated "On a criterion of relativistic invariance."] Any relativistic theory based on the notion of space-time continuum ought to satisfy the following condition. (C) If $(\mathbf{r}_1, t_1)$ and $(\mathbf{r}_2, t_2)$ are two world points which cannot be connected by a light signal, i.e., if the inequality $c|t_1-t_2|<|\mathbf{r}_1-\mathbf{r}_2|$ holds, any two events at these world points must not influence one another. The formal relativistic invariance of a theory does not necessarily imply the condition (C). Using Dirac's "many-time" formalism the author investigates, from this point of view, several possible forms of quantum electrodynamics (partly generalizing those which have appeared in the literature). In order to overcome the well-known convergence difficulties, in most of these theories a four vector λ is introduced, which is then made to tend to zero. The author shows that as long as $\lambda\neq0$ these theories do not satisfy the condition (C). [Reviewer's note. Since it is impossible to select a nonvanishing four vector in an invariant fashion, these theories cannot be considered relativistically invariant for $\lambda\neq0$ unless the components of λ are introduced as variables which describe new degrees of freedom of the system in question. So far only the limiting case $\lambda\rightarrow0$ has been assumed to be physically significant.] In the last section the author briefly discusses the "computational scheme" of Heitler and Peng [Proc. Cambridge Philos. Soc. 38, 296–312 (1942)] and Heisenberg's S-matrix [Z. Physik 120, 513–538 (1943); these Rev. 4, 292]. A fuller discussion is reserved for a later publication. [A translation appeared in Acad. Sci. USSR. J. Phys. 10, 333–340 (1946); these Rev. 8, 303.]

V. Bargmann (Princeton, N. J.).

Gustafson, Torsten. On divergencies in the theory of interaction of a nucleon with a scalar meson field. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 16, no. 2, 8 pp. (1946).

The interaction of a nucleon with a neutral scalar meson field is considered, with the additional limitation that the interaction terms contains the meson field wave-function but not its derivatives ($f=0$). In the weak-coupling approximation the Dirac wave-function of the nucleon and the wave-function of the meson field are expanded respectively as $\psi=\psi^0+g\psi^1+\cdots$ and $\Phi=\Phi^0+g\Phi^1+\cdots$. The methods of M. Riesz [C. R. Congrès Internat. Math., Oslo, 1936, vol. 2, pp. 44–45] are used to determine Φ^1. The self-energy term corresponding to the expression $\psi^{0*}\beta\psi^0\Phi^1$, for the

nucleon with respect to the meson field, is finite whereas the part arising from the real part of $\psi^{1*}\beta\psi^{4}\Phi^{0}$, i.e., depending on the vacuum-field fluctuations, diverges. In further work it is hoped to apply Riesz's methods to the solution of the Dirac equation for the nucleon wave-function.

C. Strachan (Aberdeen).

Gustafson, Torsten. On the elimination of certain divergencies in quantum electrodynamics. Ark. Mat. Astr. Fys. **34A**, no. 2, 9 pp. (1946).

This paper is a continuation of earlier ones by the author [Kungl. Fysiografiska Sällskapets i Lund Fördhandlingar [Proc. Roy. Physiog. Soc. Lund] **15**, no. 28, 277–288 (1945); Nature **157**, 734 (1946); these Rev. **7**, 180, 536; cf. the preceding review] on the calculation of the electron self-energy. The contribution to the self-energy by the forced oscillations of the electron under the action of the vacuum field fluctuation is considered. For this, Dirac's equation is solved by a method due to M. Riesz [see, for example, B. B. Baker and E. T. Copson, The Mathematical Theory of Huygens' Principle, Oxford University Press (1939); these Rev. **1**, 315]. It is shown that only photons of positive energy are necessary for the convergence of the electromagnetic self-energy to the second approximation. *C. Kikuchi.*

Nilsson, S. Bertil. On the electrostatic self-energy of the electron in the hole theory. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **16**, no. 24, 230–238 (1946).

The electrostatic part of the self-energy of an electron is considered in the second-order approximation of perturbation theory using the hole theory, i.e., the vacuum described by completely filled negative energy states [cf. Weisskopf, Physical Rev. (2) **56**, 72–85 (1939)] which, in the usual treatment, gives logarithmic divergence. The electron charge-current density vector and the potentials of the electromagnetic field are redefined in accordance with M. Riesz's method of solving hyperbolic differential equations and the electrostatic energy is obtained by analytic continuation of the appropriate expression. [Cf. the second preceding review.] The charge-current density vector is defined so that, in accordance with the hole theory, its value is zero for the vacuum. Finally, neglecting retardation effects, an energy is obtained which still diverges logarithmically. *C. Strachan* (Aberdeen).

***Riesz, Marcel. Sur certaines notions fondamentales en théorie quantique relativiste.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 123–148. Jul. Gjellerups Forlag, Copenhagen, 1947.

The author uses the Fourier integral representation of the solutions of the relativistic Schrödinger equation to obtain a relation between the integral of the normal component of the Gordon-Klein current vector over a three-dimensional subspace of space-time and the integral of the absolute value of the Fourier transform of the wave function over the subspace of the energy-momentum space with variables p^{μ} defined by $g_{\mu\nu}p^{\mu}p^{\nu}=m_0$, where m_0 is the rest mass of the particle and the velocity of light is taken to be one. A corresponding result is obtained for the solutions of the Dirac equations relating a surface integral of the Dirac current vector in space-time with the integral, over the region mentioned above of energy momentum space, of the vector formed from the Fourier transforms of the components of the wave functions in the same manner as the current vector.

It is also shown that, if the Dirac equation is generalized by replacing the simple spinor representing the wave functions by a two index spinor, then a similar result also holds if the current vector is replaced by a certain real second order tensor. The physical significance of this tensor, which can be defined for any two index spinor, is not discussed. [The reviewer wishes to point out that if the spinor is suitably restricted the tensor represents the Lorentz transformation isomorphic to the spin transformation represented by the two index spinor.] *A. H. Taub.*

***Riesz, Marcel. Éléments de probabilité en théorie quantique relativiste.** Försäkringsmatematiska Studier Tillägnade Filip Lundberg, pp. 221–222. Stockholm, 1946.

Summary of the paper reviewed above.

Kwal, Bernard. Espace spinoriel et théorie des sous-spineurs. C. R. Acad. Sci. Paris **223**, 1100–1101 (1946).

The author tries to define a space called a spinor space Σ. The points of this space are given coordinates which are linear combinations of dotted and undotted two-component spinors. Since each of these quantities has different transformation laws, this definition depends on the coordinate system in terms of which the original spinors are determined. The quantities defined in terms of the space Σ are therefore also dependent on the original coordinate system. This so-called generalization of the spinor calculus is therefore not a promising one for application to the theories of particles with spin. *A. H. Taub* (Princeton, N. J.).

Schönberg, Mario. Classical theory of the point electron. I. Summa Brasil. Math. **1**, no. 5, 41–75 (1946). (English. Portuguese summary)

The electromagnetic field created by an electron (with positive kinetic energy) is taken to be the retarded field. This field is split into two parts: the "attached" field and the "radiated" field, the former being half the sum, the latter half the difference, of the retarded and advanced fields. The assumption is introduced that the radiated field does, but the attached field does not, act on the particle by which it is created, while both radiated and attached fields of a given particle will act on any other particle. An invariant action principle is given from which the equations of motion of particles and fields are derived. The energy (which is not positive definite) and momentum of the field are finite and form a four-vector. A stationary state of motion of a system is defined as a state where each particle acts on the others but not on itself, corresponding to no radiation of energy; for such states the scheme gives an entire action-at-a-distance description. The total radiated field, on the other hand, is a field in the Faraday-Maxwell sense. Only if the motion of the electron is nonstationary is there a self-force leading to the usual expression for radiation loss.

A. Pais (Princeton, N. J.).

Schönberg, Mario. Classical theory of the point electron. II. Summa Brasil. Math. **1**, no. 6, 77–114 (1946). (English. Portuguese summary)

The scheme given in part I [see the preceding review] is worked out further. Classical stable orbits in the electromagnetic Kepler problem are indicated [the correspondence consideration in this connection, given on p. 77, is not justified]. From an analysis of the boundary conditions of the field potentials it follows that this stability is due to the compensation (in a stationary motion) of the energy

dissipated by the outgoing spherical waves of the retarded field (electron behaving as a source) by the energy gained from the ingoing spherical waves of the advanced field (electron behaving as a sink). This compensation holds rigorously only as a time average.

The states of negative kinetic energy, usually discarded in classical theory in a consistent way, are interpreted as describing "anti-particles" which create an advanced field with boundary conditions on the particle motion at $t=+\infty$. An anti-particle moves in the same way as a particle with opposite sign of charge (and positive kinetic energy) in stationary as well as nonstationary motions, the attached field for an anti-particle being defined as for an ordinary particle [see the preceding review], the radiated field as half the difference of advanced and retarded field.

A. Pais (Princeton, N. J.).

Schönberg, Mario. Classical theory of the point electron. Physical Rev. (2) **69**, 211–224 (1946).

This paper contains the main arguments and announces the results of the papers surveyed in the two preceding reviews. It is pointed out that the field generated by a particle (of either positive or negative kinetic energy) in nonstationary motion is always a retarded field for an observer whose proper time flows in the same sense as the particle's proper time, while for this same observer the radiated field of a particle (of either positive or negative kinetic energy) is always half the difference between retarded and advanced fields. *A. Pais* (Princeton, N. J.).

Critchfield, Charles L. Electron waves in the magnetic dipole field of a neutron. Physical Rev. (2) **71**, 258–267 (1947).

The Dirac equation is solved for an electron moving in the field of a central point particle ("neutron") which has a magnetic dipole moment but no electric charge. Retardation effects as well as rotational inertia are neglected. Whereas the neutron-negaton system does not exhibit bound states, such a state (of the type $P_{\frac{1}{2}}$ and with zero binding energy) is found for the neutron-positon system. In accordance with Massey [Proc. Roy. Soc. London. Ser. A. **127**, 666–670 (1930)] the scattering cross section of the electron by the neutron is found to be $\sim\pi\nu^2(e^2/Mc^2)^2$ (ν, the number of magnetons of the neutron; M, its mass). *A. Pais.*

Durand, Émile. Sur dix relations conséquences des équations du second ordre de Dirac. C. R. Acad. Sci. Paris **218**, 36–38 (1944). [MF 13450]

Banderet, Pierre Paul. Zur theorie singulärer Magnetpole. Helvetica Phys. Acta **19**, 503–522 (1946).

The existence of quantised magnetic poles of strength $\mu hc/e$, $2\mu=0, \pm1, \pm2, \cdots$, has been discussed by Dirac [Proc. Roy. Soc. London. Ser. A. **133**, 60–72 (1931)]. For an electron in the Coulomb magnetic field of such a pole Tamm [Z. Physik **71**, 141–150 (1931)], and Fierz [same Acta **17**, 27–34 (1944); these Rev. **6**, 111] have discussed the eigenfunctions of the corresponding Schrödinger equation

$$-\frac{\hbar^2}{2m}\left\{\Delta\psi-\frac{1}{r^2}\left[\frac{2i\mu}{1+\cos\theta}\frac{\partial}{\partial\phi}+\mu^2\frac{1-\cos\theta}{1+\cos\theta}\right]\psi\right\}=E\psi.$$

Here the scattering of an electron in this field is discussed in classical theory, and in nonrelativistic quantum theory using a series development in terms of generalised "spherical harmonics" which occur in the eigenfunctions.

For a charged particle in the magnetic Coulomb field the Dirac relativistic ψ-functions correspond to a representation of the rotation group with operators of angular momentum differing from the usual ones by the addition of a component parallel to the line joining magnetic pole and electric charge [Fierz, loc. cit.]. For an electron there are physically applicable wave functions. When the Dirac equation contains terms $F_{\mu\nu}\gamma^\mu\gamma^\nu\psi$ in the field strengths $F_{\mu\nu}$, giving an additional magnetic moment, and this is assumed true for the proton, such wave functions do not exist and the particle can have no stable interaction with a magnetic pole.

C. Strachan (Aberdeen).

***Pauli, Wolfgang. Meson Theory of Nuclear Forces.** Interscience Publishers, Inc., New York, 1946. 69 pp. $2.00.

This book is based on the lectures which the author gave at the Massachusetts Institute of Technology in the autumn of 1944. In some cases, however, even papers published later have been taken into account. After quoting the main experimental data on the two-nucleon problem and briefly indicating the theoretical formalism the author discusses the more important types of meson fields (charged scalar, symmetrical pseudoscalar, vector theory and various "mixtures") and the interactions between point-source nucleons derived from them by perturbation theory. Pair theories are not considered. Chapter II introduces the theory of an extended source where it is possible to define a nucleon radius, or rather a constant of dimension length^{-1} which determines the "spin inertia." The actual calculations are carried out with a neutral pseudoscalar meson field and it is shown how the assumption of large spin inertia ("strong coupling" case) leads to the existence of excited nucleon states (isobars) with higher spin values. The same formalism and field is used in chapter III for a classical treatment of the scattering of mesons by nucleons. The results are studied in some limiting cases and the corresponding calculation of Bhabha for transverse vector mesons is discussed. In a separate section the author turns to the magnetic moments of neutron and proton, pointing out that neither the strong coupling theory nor the weak coupling version (the latter being treated by the λ-limiting process of Wentzel and Dirac) can be made to conform to experimental facts. The next chapter deals with the quantum theory of scattering, including the theory of radiation damping usually ascribed to Heitler [the reviewer would like to remark that this theory was developed earlier by I. Waller, Z. Physik **88**, 436–448 (1934)] and Heisenberg's theory of "observable quantities," best known as the S-matrix theory.

Chapter V gives a brief outline of the theory of neutron-proton scattering based on "weak coupling" and brings out the discrepancy between the exchange theory of nuclear interaction and the experimental results of Amaldi and co-workers [however, recent experiments by Powell and Occhialini, communicated at the Physical Society Conference at Cambridge, England, in July, 1946, have not confirmed the forward scattering effect found by the Rome group]. The last chapter is devoted to the strong coupling theory of the two-nucleon system. Since the book was written this theory has lost some of its interest as it has turned out to be incompatible with the fact that the ^1S-state of the deuteron is virtual [this was reported by Wentzel at the conference mentioned above; cf. F. Villars, Helvetica Phys. Acta **19**, 323–354 (1946); these Rev. **8**, 304]. In conclusion the author remarks that the most hopeful approach to the problem of nuclear forces seems to be the nonrelativistic theory with nucleons of finite size. *L. Hulthén.*

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